The Fitzhugh-Nagamo model for the electrical impulse in a neuron states that, in the absense of relaxation effects, the electrical potential in a neuron $v(t)$ obeys the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -v\left[v^2 - (1+a)\,v + a\right]$$

where a is a positive constant such that $0 < a < 1$.[1]

(a) For what values of v is v unchanging (that is, $\mathrm{d}v/\mathrm{d}t = 0$)?

Note that the given differential equation *is* the derivative, so we solve

$$0 = -v\left[v^2 - (1+a)\,v + a\right]$$

so

$$0 = v \quad \text{or} \quad 0 = v^2 - (1+a)\,v + a$$

Using the quadratic formula, we get

$$\begin{aligned}
0 = v \quad &\text{or} \quad v = \frac{-(-(1+a)) \pm \sqrt{(-(1+a))^2 - 4\,(1)\,(a)}}{2\,(1)} \\
0 = v \quad &\text{or} \quad v = \frac{1+a \pm \sqrt{(1+a)^2 - 4a}}{2} \\
0 = v \quad &\text{or} \quad v = \frac{1+a \pm \sqrt{1+2a+a^2-4a}}{2} \\
0 = v \quad &\text{or} \quad v = \frac{1+a \pm \sqrt{1-2a+a^2}}{2} \\
0 = v \quad &\text{or} \quad v = \frac{1+a \pm \sqrt{(1-a)^2}}{2}
\end{aligned}$$

and since $1 - a > 0$, we have

$$\begin{aligned}
0 = v \quad &\text{or} \quad v = \frac{1+a+(1-a)}{2} \quad \text{or} \quad v = \frac{1+a-(1-a)}{2} \\
0 = v \quad &\text{or} \quad v = 1 \quad \text{or} \quad v = a
\end{aligned}$$

(b) For what values of v is v increasing?

v is increasing if $\mathrm{d}v/\mathrm{d}t > 0$.

$$\begin{aligned}
-v\left[v^2 - (1+a)\,v + a\right] &> 0 \\
-v\left(v^2 - v - av + a\right) &> 0 \\
-v\,(v\,(v-1) - (av - a)) &> 0 \\
-v\,(v\,(v-1) - a\,(v-1)) &> 0 \\
-v\,(v-1)\,(v-a) &> 0
\end{aligned}$$

Since $0 < a < 1$, the above inequality is true if $v < 0$ or if $a < v < 1$. Thus, v is increasing on $(-\infty,0) \cup (a,1)$.

[1] Stewart, *Calculus, Early Transcendentals*, p. 590, #10.

(c) For what values of v is v decreasing?

Using our analysis from part (b), we see that v is decreasing on $(0,a) \cup (1,\infty)$.