Solve the differential equation.[1]

$$\frac{\mathrm{d}H}{\mathrm{d}R} = \frac{RH^2\sqrt{1+R^2}}{\ln(H)}$$

We separate H and R, and then integrate.

$$\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}R} &= \frac{RH^2\sqrt{1+R^2}}{\ln(H)} \\
\frac{\ln(H)\mathrm{d}R}{H^2} \cdot \frac{\mathrm{d}H}{\mathrm{d}R} &= \frac{RH^2\sqrt{1+R^2}}{\ln(H)} \cdot \frac{\ln(H)\mathrm{d}R}{H^2} \\
\frac{\ln(H)}{H^2}\,\mathrm{d}H &= R\sqrt{1+R^2}\,\mathrm{d}R \\
\int \frac{\ln(H)}{H^2}\,\mathrm{d}H &= \int R\sqrt{1+R^2}\,\mathrm{d}R
\end{aligned}$$

For the right side, we let $u = 1 + R^2$, so $\mathrm{d}u = 2R\,\mathrm{d}R$. We get

$$\begin{aligned}
\int R\sqrt{1+R^2}\,\mathrm{d}R &= \frac{1}{2}\int 2R\sqrt{1+R^2}\,\mathrm{d}R \\
&= \frac{1}{2}\int \sqrt{u}\,\mathrm{d}u \\
&= \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C \\
&= \frac{1}{3}\left(1+R^2\right)^{3/2} + C
\end{aligned}$$

For the left side, following a couiple of unsuccessful attempts at substitution, we try integration by parts. We let $u = \ln(H)$ so $\mathrm{d}v = H^{-2}\,\mathrm{d}H$. Then $\mathrm{d}u = \frac{1}{H}\,\mathrm{d}H$ and $v = -H^{-1}$.

$$\begin{aligned}
\int \frac{\ln(H)}{H^2}\,\mathrm{d}H &= -\frac{\ln(H)}{H} - \int -\frac{1}{H}\cdot\frac{1}{H}\,\mathrm{d}H \\
&= -\frac{\ln(H)}{H} - \int -H^{-2}\,\mathrm{d}H \\
&= -\frac{\ln(H)}{H} - \left(H^{-1}\right) \\
&= -\frac{\ln(H)}{H} - \frac{1}{H}
\end{aligned}$$

(We do not need a constant, because we already have one from the right side.)

Substituting back into our equation, we have

$$\begin{aligned}
\int \frac{\ln(H)}{H^2}\,\mathrm{d}H &= \int R\sqrt{1+R^2}\,\mathrm{d}R \\
-\frac{\ln(H)}{H} - \frac{1}{H} &= \frac{1}{3}\left(1+R^2\right)^{3/2} + C \\
-\frac{1+\ln(H)}{H} &= \frac{1}{3}\left(1+R^2\right)^{3/2} + C
\end{aligned}$$

We are unable to solve explicitly for H, so this gives H as an implicit function of R.

[1] Stewart, *Calculus, Early Transcendentals*, p. 605, #8.