Calculus II, Section 9.3, #16
Separable Equations

Find the solution of the differential equation that satisfies the given initial condition.[1]

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \sqrt{Pt}, \qquad P(1) = 2$$

We separate P and t, and integrate.

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}t} &= \sqrt{Pt} \\
\frac{\mathrm{d}P}{\mathrm{d}t} &= \sqrt{P}\sqrt{t} \\
\frac{1}{\sqrt{P}}\,\mathrm{d}P &= \sqrt{t}\,\mathrm{d}t \\
\int P^{-1/2}\,\mathrm{d}P &= \int t^{1/2}\,\mathrm{d}t \\
\frac{P^{1/2}}{\frac{1}{2}} &= \frac{t^{3/2}}{\frac{3}{2}} + C
\end{aligned}$$

> We actually have
>
> $$\begin{aligned}
\int P^{-1/2}\,\mathrm{d}P &= \int t^{1/2}\,\mathrm{d}t \\
\frac{P^{1/2}}{\frac{1}{2}} + C_1 &= \frac{t^{3/2}}{\frac{3}{2}} + C_2 \\
\frac{P^{1/2}}{\frac{1}{2}} &= \frac{t^{3/2}}{\frac{3}{2}} + C_2 - C_1
\end{aligned}$$
>
> But since $C_2 - C_1$ is just another constant, we write
>
> $$\frac{P^{1/2}}{\frac{1}{2}} = \frac{t^{3/2}}{\frac{3}{2}} + C$$

So the general solution (in implicit form) is

$$2\sqrt{P} = \frac{2}{3}\sqrt{t^3} + C$$

From the initial condition $P(1) = 2$, we have $P = 2$ when $t = 1$. Substituting and solving for C, we get

$$\begin{aligned}
2\sqrt{2} &= \frac{2}{3}\sqrt{1^3} + C \\
2\sqrt{2} &= \frac{2}{3} + C \\
2\sqrt{2} - \frac{2}{3} &= C
\end{aligned}$$

Substituting this value for C into our general solution

$$\begin{aligned}
2\sqrt{P} &= \frac{2}{3}\sqrt{t^3} + 2\sqrt{2} - \frac{2}{3} \\
\sqrt{P} &= \frac{1}{3}\sqrt{t^3} + \sqrt{2} - \frac{1}{3} \\
P &= \left(\frac{\sqrt{t^3} + 3\sqrt{2} - 1}{3}\right)^2
\end{aligned}$$

Thus the particular solution for $\frac{\mathrm{d}P}{\mathrm{d}t} = \sqrt{Pt}$ with the initial condition $P(1) = 2$ is

$$P(t) = \frac{1}{9}\left(\sqrt{t^3} + 3\sqrt{2} - 1\right)^2$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 605, #16.