**Calculus II, Section 9.3, #48**
**Separable Equations**

*A tank contains* $1000\,\mathrm{L}$ *of pure water. Brine that contains* $0.05\,\mathrm{kg}$ *of salt per liter of water enters the tank at the rate of* $5\,\mathrm{L/min}$. *Brine that contains* $0.04\,\mathrm{kg}$ *of salt per liter of water enters the tank at the rate of* $10\,\mathrm{L/min}$. *The solution is kept thoroughly mixed and drains from the tank at a rate of* $15\,\mathrm{l/min}$. *How much salt is in the tank (a) after* $t$ *minutes and (b) after one hour?*[1]

We want a function that gives the amount of salt at time $t$, but all the information is about how the amount of salt is changing.

Let $s(t) =$ amount, in kg of salt at time $t$. Then we have

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}t} &= (\text{rate of salt into tank}) - (\text{rate of salt out of tank}) \\ &= (0.05\,\mathrm{kg/L} \cdot 5\,\mathrm{L/min}) + (0.04\,\mathrm{kg/L} \cdot 10\,\mathrm{L/min}) - \left(\frac{s\,\mathrm{kg}}{1000\,\mathrm{L}} \cdot 15\,\mathrm{L/min}\right) \\ &= 0.25\,\mathrm{kg/min} + 0.4\,\mathrm{kg/min} - \frac{15s}{1000}\,\mathrm{kg/min}\end{aligned}$$

So we get the differential equation

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}t} &= 0.65 - \frac{15s}{1000} \\ &= \frac{65}{100} - \frac{15s}{1000} \\ \frac{\mathrm{d}s}{\mathrm{d}t} &= \frac{130 - 3s}{200}\end{aligned}$$

We separate $s$ and $t$ to get

$$\frac{1}{130 - 3s}\,\mathrm{d}s = \frac{1}{200}\,\mathrm{d}t$$

Integrate

$$\begin{aligned}\int \frac{1}{130 - 3s}\,\mathrm{d}s &= \int \frac{1}{200}\,\mathrm{d}t \\ -\frac{1}{3} \cdot \ln|130 - 3s| &= \frac{1}{200}t + C_1 \\ \ln|130 - 3s| &= -\frac{3}{200}t + C_2 \\ |130 - 3s| &= \mathrm{e}^{-\frac{3}{200}t + C_2} \\ |130 - 3s| &= C_3\mathrm{e}^{-3t/200} \\ 130 - 3s &= C_4\mathrm{e}^{-3t/200} \\ -3s &= -130 + C_4\mathrm{e}^{-3t/200} \\ s &= \frac{130 - C_4\mathrm{e}^{-3t/200}}{3}\end{aligned}$$

Since we begin with pure water, we have $s(0) = 0$. Substituting,

$$\begin{aligned}0 &= \frac{130 - C_4\mathrm{e}^{-3\cdot 0/200}}{3} \\ 0 &= 130 - C_4 \\ C_4 &= 130\end{aligned}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 607, #48.

So our function is

$$s(t) = \frac{130 - 130\mathrm{e}^{-3t/200}}{3}$$

After one hour (60 min), we have

$$s(60) = \frac{130 - 130\mathrm{e}^{-3\cdot 60/200}}{3}$$
$$s(60) \approx 25.7153$$

Thus, after one hour there is about 25.72 kg of salt in the tank.