According to Newton's Law of Universal Gravitation, the gravitational force on an object of mass m that has been projected vertically upward from the earth's surface is

$$F = \frac{mgR^2}{(x+R)^2}$$

where $x = x(t)$ is the object's distance above the surface at time t, R is the earth's radius, and g is the acceleration due to gravity. Also, by Newton's Second Law, $F = ma = m\,(\mathrm{d}v/\mathrm{d}t)$ and so[1]

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{mgR^2}{(x+R)^2}$$

(a) *Suppose a rocket is fired vertically upward with an initial velocity v_0. Let h be the maximum height above the surface reached by the object. Show that*

$$v_0 = \sqrt{\frac{2gRh}{R+h}}$$

(Hint: By the Chain Rule, $m\,(\mathrm{d}v/\mathrm{d}t) = mv\,(\mathrm{d}v/\mathrm{d}x)$.)

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = m\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

Since $v = \frac{\mathrm{d}v}{\mathrm{d}x}$, we have

$$-\frac{mgR^2}{(x+R)^2} = mv\frac{\mathrm{d}v}{\mathrm{d}x}$$
$$-\frac{gR^2}{(x+R)^2} = v\frac{\mathrm{d}v}{\mathrm{d}x}$$

Separate v and x, and integrate.

$$-\frac{gR^2}{(x+R)^2}\,\mathrm{d}x = v\,\mathrm{d}v$$
$$\int -\frac{gR^2}{(x+R)^2}\,\mathrm{d}x = \int v\,\mathrm{d}v$$

So we get the general solution

$$\frac{gR^2}{x+R} = \frac{v^2}{2} + C$$

When $x = 0$, $v = v_0$. Substituting,

$$\frac{gR^2}{0+R} = \frac{v_0^2}{2} + C$$
$$gR - \frac{v_0^2}{2} = C$$

Substituting this into our general solution gives the particular solution

$$\frac{gR^2}{x+R} = \frac{v^2}{2} + gR - \frac{v_0^2}{2}$$

[1]Stewart, *Calculus, Early Transcendentals*, p. 607, #54.

$$\begin{aligned}\frac{v_0^2}{2} &= \frac{v^2}{2} + gR - \frac{gR^2}{x+R} \\ v_0^2 &= v^2 + 2gR - \frac{2gR^2}{x+R}\end{aligned}$$

When the rocket reaches the top of its trajectory, $x = h$ and $v = 0$. Substituting,

$$\begin{aligned}v_0^2 &= 0^2 + 2gR - \frac{2gR^2}{h+R} \\ v_0^2 &= 2gR - \frac{2gR^2}{h+R} \\ v_0^2 &= \frac{2gR\,(h+R)}{h+R} - \frac{2gR^2}{h+R} \\ v_0^2 &= \frac{2gRh + 2gR^2 - 2gR^2}{h+R} \\ v_0^2 &= \frac{2gRh}{h+R} \\ \sqrt{v_0^2} &= \sqrt{\frac{2gRh}{h+R}}\end{aligned}$$

Since $v_0 > 0$,

$$v_0 = \sqrt{\frac{2gRh}{h+R}}$$

(b) *Calculate $v_e = \lim_{h\to\infty} v_0$. This limit is called the **escape velocity** for the earth.*

$$\begin{aligned}v_e &= \lim_{h\to\infty} v_0 \\ &= \lim_{h\to\infty} \sqrt{\frac{2gRh}{h+R}} \\ &= \lim_{h\to\infty} \sqrt{\frac{\frac{2gRh}{h}}{\frac{h+R}{h}}} \\ &= \lim_{h\to\infty} \sqrt{\frac{2gR}{1+\frac{R}{h}}}\end{aligned}$$

As $h \to \infty$, $\frac{R}{h} \to 0$, so

$$v_e = \sqrt{2gR}$$

(c) *Use $R = 3960\,\text{mi}$ and $g = 32\,\text{ft/s}^2$ to calculate v_e in feet per second and miles per second.*

$$R = 3960\,\text{mi} = 20{,}908{,}800\,\text{ft}$$

so

$$\begin{aligned}v_e &= \sqrt{2 \cdot 32 \cdot 20{,}908{,}800} \\ &\approx 36580.91\,\text{ft/s} \\ &\approx 6.93\,\text{mi/s} \\ &\approx 24{,}941.53\,\text{mi/h}\end{aligned}$$