Calculus II, Section 9.6, #4
Predator-Prey Systems

Lynx eat snowshoe hares and snowshoe hares eat woody plants like willows. Suppose that, in the absence of hares, the willow population will grow exponentially and the lynx population will decay exponentially. In the absence of lynx and willow, the hare population will decay exponentially. If $L(t)$, $H(t)$, and $W(t)$ represent the populations of these three species at time t, write a system of differential equations as a model for their dynamics. If the constants in your equation are all positive, explain why you have used plus or minus signs.[1]

We have three interacting populations: lynx $L(t)$, hares $H(t)$, and willows $W(t)$. Let's examine each sentence.

"...in the absence of hares, the willow population will grow exponentially ..." indicates $\frac{\mathrm{d}W}{\mathrm{d}t} = k_1 W$, where k_1 would be a positive constant. To emphasize that $\frac{\mathrm{d}W}{\mathrm{d}t} = k_1 W$ represents growth of the willow population, we'll write $\frac{\mathrm{d}W}{\mathrm{d}t} = +k_1 W$.

"...in the absence of hares, ...the lynx population will decay exponentially." indicates that $\frac{\mathrm{d}L}{\mathrm{d}t} = -k_2 L$, where k_2 is a positive constant and the negative sign indicates decay.

"In the absence of lynx and willow, the hare population will decay exponentially." indicates that $\frac{\mathrm{d}H}{\mathrm{d}t} = -k_3 H$, where k_3 is a positive constant and the negative sign indicates decay.

"Lynx eat snowshoe hares and snowshoe hares eat woody plants like willows." indicates that, given encounters between the three species, lynx win (they're eating hares, but nothing eats lynx), hares lose and win (hares lose when eaten by lynx, but win when they eat willows), and willows lose (willows get eaten by the hares). In terms of the derivatives, we get $\frac{\mathrm{d}L}{\mathrm{d}t} = +c_1 LH$ where c_1 is a positive constant and the + indicates that the lynx population grows from the encounter, $\frac{\mathrm{d}H}{\mathrm{d}t} = -c_2 LH + c_3 HW$ where c_2 and c_3 are positive constants and the − indicates that hares lose when they encounter lynx and the + indicates hares win when they encounter willows, and $\frac{\mathrm{d}W}{\mathrm{d}t} = -c_4 HW$ where c_4 is a positive constant and the − indicates that the willows lose when they encounter hares.

Combining the equations for $\frac{\mathrm{d}L}{\mathrm{d}t}$, $\frac{\mathrm{d}H}{\mathrm{d}t}$, and $\frac{\mathrm{d}W}{\mathrm{d}t}$, we get the system of differential equations

$$\begin{array}{rcrcrcl}
\frac{\mathrm{d}L}{\mathrm{d}t} & = & -k_2 L & + & c_1 LH & & \\
\frac{\mathrm{d}H}{\mathrm{d}t} & = & -k_3 H & - & c_2 LH & + & c_3 HW \\
\frac{\mathrm{d}W}{\mathrm{d}t} & = & +k_1 W & & & - & c_4 HW
\end{array}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 632, #4.