Populations of aphids and ladybugs are modeled by the equations[1]

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2A - 0.01AL$$
$$\frac{\mathrm{d}L}{\mathrm{d}t} = -0.5L + 0.0001AL$$

(a) Find the equilibrium solutions and explain their significance.

We solve

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2A - 0.01AL = 0$$
$$\frac{\mathrm{d}L}{\mathrm{d}t} = -0.5L + 0.0001AL = 0$$

or

$$A\,(2 - 0.01L) = 0$$
$$L\,(-0.5 + 0.0001A) = 0$$

So $A = 0$ and $L = 0$ is an equilibrium solution. This makes sense, because if there are no aphids and no ladybugs, the populations won't grow. We also get

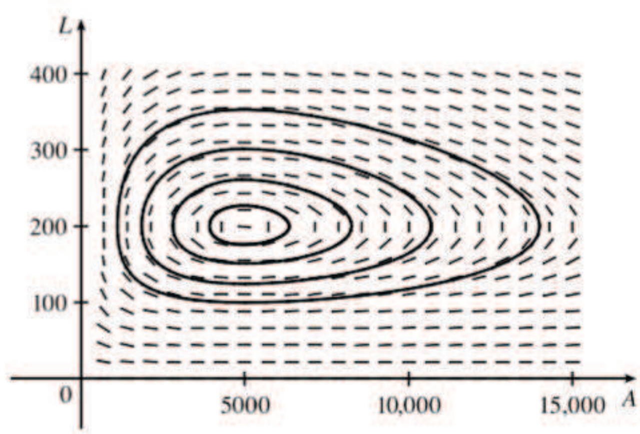
$$2 - 0.01L = 0$$
$$-0.5 + 0.0001A = 0$$

So $A = 5000$ and $L = 200$ is also an equilibrium solution. This tells us that a population of 5000 aphids and 200 ladybugs is stable.

(b) Find an expression for $\mathrm{d}L/\mathrm{d}A$.

$$\frac{\mathrm{d}L}{\mathrm{d}A} = \frac{\frac{\mathrm{d}L}{\mathrm{d}t}}{\frac{\mathrm{d}A}{\mathrm{d}t}} = \frac{-0.5L + 0.0001AL}{2 - 0.01AL}$$

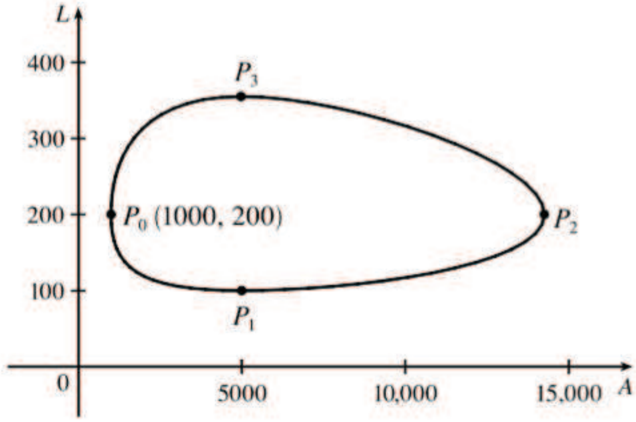
(c) The direction field for the differential equation in part (b) is shown. Use it to sketch a phase portrait. What do the phase trajectories have in common?



Note that all of the phase trajectories seem to be closed trajectories with the equilibrium point $A = 5000$ and $L = 200$ inside the trajectory.

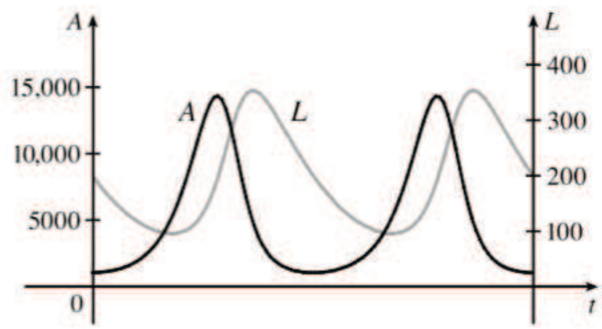
[1] Stewart, *Calculus, Early Transcendentals*, p. 633, #10.

(d) Suppose that at time $t = 0$ there are 1000 aphids and 200 ladybugs. Draw the corresponding phase trajectory and use it to describe how both populations change.



At point P_0, the populations are 1000 aphids and 200 ladybugs. As we move towards P_1, the population of ladybugs is decreasing, so the population of aphids is increasing, slowly at first, and then more rapidly as we approach P_1. The population of aphids continues to increase, and the population of ladybugs also begins to increase, slowly near P_1 but then more rapidly near P_2. The number of aphids then begins to decrease rapidly as the number of ladybugs increases as we near P_3. Finally, both populations decrease as we move back towards P_0.

(e) Use part (d) to make rough sketches of the aphid and ladybug populations as functions of t. How are the graphs related to each other?



The periods of $A(t)$ and $L(t)$ seem to be the same, with the cycle of $L(t)$ about 1/4 of a cycle behind that of $A(t)$.