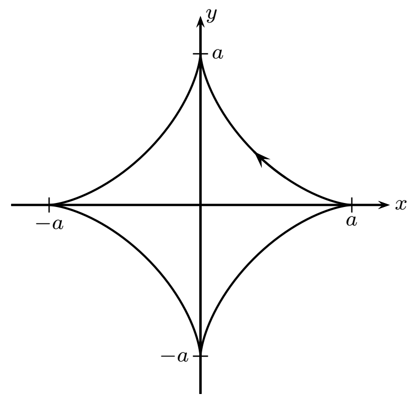
Find the area enclosed by the astroid $x = a\cos^3(\theta)$, $y = a\sin^3(\theta)$.[1]



The desired area, A, traced out for $0 \le \theta \le 2\pi$, is shown at right. Note the orientation in the first quadrant. In terms of x and y, the first quadrant area is given by

$$\int_{x=0}^{x=a} y \, \mathrm{d}x$$

but the curve is traced from $\theta = 0$ to $\theta = \pi/2$, we have

$$A = 4\int_{\theta=\pi/2}^{\theta=0} y \, \mathrm{d}x$$

Since $y = a\sin^3(\theta)$ and $\mathrm{d}x = 3a\cos^2(\theta) \cdot -\sin(\theta)\, \mathrm{d}\theta$, we get

$$\begin{aligned} &= 4\int_{\theta=\pi/2}^{\theta=0} a\sin^3(\theta)\ 3a\cos^2(\theta) \cdot -\sin(\theta)\, \mathrm{d}\theta \\ &= -12a^2 \int_{\theta=\pi/2}^{\theta=0} \sin^4(\theta)\cos^2(\theta)\, \mathrm{d}\theta \end{aligned}$$

Let's find the indefinite integral

$$\int \sin^4(\theta)\cos^2(\theta)\, \mathrm{d}\theta$$

and then use the result to evaluate the definite integral giving the area.

$$\begin{aligned} \int \sin^4(\theta)\cos^2(\theta)\, \mathrm{d}\theta &= \int \sin^2(\theta)\sin^2(\theta)\cos^2(\theta)\, \mathrm{d}\theta \\ &= \int \sin^2(\theta)\left(\sin(\theta)\cos(\theta)\right)^2\, \mathrm{d}\theta \\ &= \int \sin^2(\theta)\left(\frac{1}{2}\sin(2\theta)\right)^2\, \mathrm{d}\theta \\ &= \int \sin^2(\theta) \cdot \frac{1}{4}\sin^2(2\theta)\, \mathrm{d}\theta \\ &= \int \frac{1}{2}\left(1 - \cos(2\theta)\right) \cdot \frac{1}{4}\sin^2(2\theta)\, \mathrm{d}\theta \\ &= \frac{1}{8}\int \sin^2(2\theta) - \cos(2\theta)\sin^2(2\theta)\, \mathrm{d}\theta \\ &= \frac{1}{8}\int \frac{1}{2}\left(1 - \cos(4\theta)\right)\, \mathrm{d}\theta - \frac{1}{8}\int \cos(2\theta)\sin^2(2\theta)\, \mathrm{d}\theta \end{aligned}$$

For the second integral, we let $u = \sin(2\theta)$, so $\mathrm{d}u = 2\cos(2\theta)\, \mathrm{d}\theta$. Substituting,

$$= \frac{1}{16}\int 1 - \cos(4\theta)\, \mathrm{d}\theta - \frac{1}{16}\int u^2\, \mathrm{d}u$$

[1]Stewart, *Calculus, Early Transcendentals*, p. 655, #34.

$$\begin{aligned}
&= \frac{1}{16}\left[\theta - \frac{1}{4}\sin(4\theta)\right] - \frac{1}{16}\cdot\frac{u^3}{3} \\
&= \frac{1}{16}\theta - \frac{1}{64}\sin(4\theta) - \frac{1}{48}\sin^3(2\theta)
\end{aligned}$$

Now we evaluate

$$\begin{aligned}
A &= -12a^2 \int_{\theta=\pi/2}^{\theta=0} \sin^4(\theta)\cos^2(\theta)\,\mathrm{d}\theta \\
&= -12a^2 \left[\frac{1}{16}\theta - \frac{1}{64}\sin(4\theta) - \frac{1}{48}\sin^3(2\theta)\right]_{\theta=\pi/2}^{\theta=0} \\
&= -12a^2 \left[\left(\frac{1}{16}\cdot 0 - \frac{1}{64}\sin(4\cdot 0) - \frac{1}{48}\sin^3(2\cdot 0)\right)\right. \\
&\qquad \left. - \left(\frac{1}{16}\cdot\frac{\pi}{2} - \frac{1}{64}\sin\left(4\cdot\frac{\pi}{2}\right) - \frac{1}{48}\sin^3\left(2\cdot\frac{\pi}{2}\right)\right)\right] \\
&= -12a^2\left[(0) - \left(\frac{\pi}{32} - 0 - 0\right)\right] \\
&= -12a^2 \cdot -\frac{\pi}{32} \\
&= \frac{3a^2\pi}{8}
\end{aligned}$$

Thus, the area of the astroid is $\frac{3a^2\pi}{8}$.