Find the points on the given curve where the tangent line is horizontal or vertical.[1]

$$r = \mathrm{e}^{\theta}$$

If $r = \mathrm{e}^{\theta}$ then

$$x = r\cos(\theta) = \mathrm{e}^{\theta}\cos(\theta)$$

so

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}\theta} &= \mathrm{e}^{\theta} \cdot -\sin(\theta) + \cos(\theta) \cdot \mathrm{e}^{\theta} \\ &= \mathrm{e}^{\theta}\left(\cos(\theta) - \sin(\theta)\right)\end{aligned}$$

and

$$y = r\sin(\theta) = \mathrm{e}^{\theta}\sin(\theta)$$

so

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}\theta} &= \mathrm{e}^{\theta} \cdot +\cos(\theta) + \sin(\theta) \cdot \mathrm{e}^{\theta} \\ &= \mathrm{e}^{\theta}\left(\cos(\theta) + \sin(\theta)\right)\end{aligned}$$

From our work with parametric curves, we know

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}}$$

and there is a horizontal tangent wherever $\mathrm{d}y/\mathrm{d}\theta = 0$ provided $\mathrm{d}x/\mathrm{d}\theta \neq 0$. So we solve

$$0 = \mathrm{e}^{\theta}\left(\cos(\theta) + \sin(\theta)\right)$$

Since $\mathrm{e}^{\theta} \neq 0$ for any θ, we have

$$\begin{aligned}0 &= \cos(\theta) + \sin(\theta) \\ \sin(\theta) &= -\cos(\theta) \\ \tan(\theta) &- 1\end{aligned}$$

or

$$\theta = \frac{3\pi}{4} + k\pi, \quad k \text{ an integer}$$

Since $\cos(\theta)$ and $\sin(\theta)$ have opposite signs in the second and fourth quadrants, $\mathrm{d}x/\mathrm{d}\theta \neq 0$ for these values of θ, and thus there are horizontal tangents at the points $\left(\mathrm{e}^{\frac{3\pi}{4}+k\pi}, \frac{3\pi}{4} + k\pi\right) \Rightarrow \left(\mp\frac{\sqrt{2}}{2}\mathrm{e}^{3\pi/4+k\pi}, \pm\frac{\sqrt{2}}{2}\mathrm{e}^{3\pi/4+k\pi}\right)$.

We also know there is a vertical tangent wherever $\mathrm{d}x/\mathrm{d}\theta = 0$ provided $\mathrm{d}y/\mathrm{d}\theta \neq 0$. So we solve

$$0 = \mathrm{e}^{\theta}\left(\cos(\theta) - \sin(\theta)\right)$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 667, #64.

Since $e^{\theta} \neq 0$ for any θ, we have

$$\begin{aligned} 0 &= \cos(\theta) - \sin(\theta) \\ \sin(\theta) &= \cos(\theta) \\ \tan(\theta) &= 1 \end{aligned}$$

or

$$\theta = \frac{\pi}{4} + k\pi, \quad k \text{ an integer}$$

Since $\cos(\theta)$ and $\sin(\theta)$ have the same signs in the first and third quadrants, $\mathrm{d}y/\mathrm{d}\theta \neq 0$ for these values of θ, and thus there are horizontal tangents at the points $\left(e^{\frac{\pi}{4}+k\pi}, \frac{\pi}{4}+k\pi\right) \Rightarrow \left(\pm\frac{\sqrt{2}}{2}e^{\pi/4+k\pi}, \pm\frac{\sqrt{2}}{2}e^{\pi/4+k\pi}\right)$.

Here are two views of the graph of $r = e^{\theta}$ for $\theta > 0$.