Let[1]

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

(a) *Show that if* $0 \le a < b$, *then*

$$\frac{b^{n+1} - a^{n+1}}{b - a} < (n+1)\, b^n$$

Note that

$$b^{n+1} - a^{n+1} = (b-a)\left(b^n + b^{n-1}a + b^{n-2}a^2 + b^{n-3}a^3 + \cdots + ba^{n-1} + a^n\right)$$

so

$$\begin{aligned}\frac{b^{n+1} - a^{n+1}}{b - a} &= \frac{(b-a)\left(b^n + b^{n-1}a + b^{n-2}a^2 + b^{n-3}a^3 + \cdots + ba^{n-1} + a^n\right)}{b-a} \\ &= b^n + b^{n-1}a + b^{n-2}a^2 + b^{n-3}a^3 + \cdots + ba^{n-1} + a^n\end{aligned}$$

and since $a < b$

$$\begin{aligned}&< b^n + b^{n-1} \cdot b + b^{n-2} \cdot b^2 + b^{n-3} \cdot b^3 + \cdots + b \cdot b^{n-1} + b^n \\ &= b^n + b^n + b^n + \cdots + b^n + b^n \\ &= (n+1)\, b^n\end{aligned}$$

Thus, $\dfrac{b^{n+1} - a^{n+1}}{b - a} < (n+1)\, b^n$.

(b) *Deduce that* $b^n\left[(n+1)\, a - nb\right] < a^{n+1}$.

From part (a) we have

$$\frac{b^{n+1} - a^{n+1}}{b - a} < (n+1)\, b^n$$

Since $b - a > 0$, we have

$$b^{n+1} - a^{n+1} < (n+1)\, b^n\, (b-a)$$

so

$$\begin{aligned}b^{n+1} - (n+1)\, b^n\, (b-a) &< a^{n+1} \\ b^{n+1} - (n+1)\, b^{n+1} + (n+1)\, ab^n &< a^{n+1} \\ b^n\left[b - (n+1)\, b + (n+1)\right] &< a^{n+1}\end{aligned}$$

Thus

$$b^n\left[(n+1)\, a - nb\right] < a^{n+1}$$

[1]Stewart, *Calculus, Early Transcendentals*, p. 706, #90.

(c) *Use* $a = 1 + 1/(n+1)$ *and* $b = 1 + 1/n$ *in part* (b) *to show that* $\{a_n\}$ *is increasing.*

Substituting the given values

$$\begin{aligned} b^n \left[(n+1) \cdot \left(1 + \frac{1}{n+1}\right) - n \cdot \left(1 + \frac{1}{n}\right) \right] &< a^{n+1} \\ b^n \left[(n+1) + 1 - n - 1 \right] &< a^{n+1} \\ b^n \left[(n+1) + 1 - n - 1 \right] &< a^{n+1} \end{aligned}$$

so

$$b^n < a^{n+1}$$

or

$$b^n = \left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1} = a^{n+1}$$

Thus the sequence is increasing.

(d) *Use* $a = 1$ *and* $b = 1 + 1/(2n)$ *in part* (b) *to show* $a_{2n} < 4$.

From part(b) we have

$$b^n \left[(n+1) a - nb \right] < a^{n+1}$$

Substituting the suggested values gives

$$\begin{aligned} \left(1 + \frac{1}{2n}\right)^n \left[(n+1) \cdot 1 - n \left(1 + \frac{1}{2n}\right) \right] &< 1^{n+1} \\ \left(1 + \frac{1}{2n}\right)^n \cdot \frac{1}{2} &< 1 \\ \left(1 + \frac{1}{2n}\right)^n &< 2 \\ \left(\left(1 + \frac{1}{2n}\right)^n\right)^2 &< 2^2 \\ \left(1 + \frac{1}{2n}\right)^{2n} &< 4 \end{aligned}$$

or

$$a_{2n} < 4$$

(e) *Use parts* (c) *and* (d) *to show that* $a_n < 4$.

Since the sequnece is increasing (part (c)), we know

$$a_n < a_{2n}$$

and from part (b)

$$a_{2n} < 4$$

Thus we get

$$a_n < a_{2n} < 4$$

(f) *Use the Monotonic Sequence Theorem to show that* $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ *exists.*

In part (c), we demonstrated that the sequence is increasing. In part (d), we demonstrated that the sequence is bounded above by 4. Thus, by the Monotonic Sequence Theorem, we know the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ has a limit.

Note: From our earlier (Calc I) work with the derivatives of logarithmic functions, we know

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \mathrm{e}$$