After injection of a dose D of insulin, the concentration of insulin in a patient's system decays exponentially and so it can be written as De^{-at}, where t represents the time in hours and a is a positive constant.[1]

(a) *If a dose D is injected every T hours, write an expression for the sum of the residual concentrations just before the $(n+1)$st injection.*

Just before the second injection, the residual concentration is De^{-aT}, since the injection is given every T hours. Just before the third injection, the residual concentration is De^{-a2T}. Following this pattern, the residual concentration before the $(n+1)$st injection is De^{-anT}. The sum of the residual concentrations is

$$De^{-aT} + De^{-a2T} + \cdots + De^{-anT}$$

This is a finite geometric series with $r = e^{-aT}$ and $a_1 = De^{-aT}$, so the sum is

$$\frac{De^{-aT}\left(1 - e^{-anT}\right)}{1 - e^{-aT}}$$

(b) *Determine the limiting pre-injection concentration.*

We calculate

$$\begin{aligned}
&\lim_{n\to\infty} \frac{De^{-aT}\left(1 - e^{-anT}\right)}{1 - e^{-aT}} \\
&\quad = \lim_{n\to\infty} \frac{\frac{De^{-aT}\left(1-e^{-anT}\right)}{e^{-aT}}}{\frac{1-e^{-aT}}{e^{-aT}}} \\
&\quad = \lim_{n\to\infty} \frac{D\left(1 - e^{-anT}\right)}{\frac{1}{e^{-aT}} - 1} \\
&\quad = \lim_{n\to\infty} \frac{D\left(1 - \frac{1}{e^{anT}}\right)}{e^{aT} - 1} \\
&\quad = \frac{D\,(1-0)}{e^{aT} - 1} \\
&\quad = \frac{D}{e^{aT} - 1}
\end{aligned}$$

(c) *If the concentration of insulin must always remain at or above a critical value C, determine a minimal dosage D in terms of C, a, and T.*

We need

$$\begin{aligned}
\frac{D}{e^{aT} - 1} &\geq C \\
D &\geq C\left(e^{aT} - 1\right)
\end{aligned}$$

So the minimal dosage is $D = C\left(e^{aT} - 1\right)$.

[1] Stewart, *Calculus, Early Transcendentals*, p. 717, #72.