Determine whether the series is convergent or divergent.[1]

$$\sum_{n=3}^{\infty} \frac{3n-4}{n^2-2n}$$

We suspect divergence because the series has the form $\sum \frac{3n}{n^2} = \sum \frac{3}{n}$, which has the form of a divergent p-series.

However, $\lim_{n\to\infty} \frac{3n-4}{n^2-2n} = \lim_{n\to\infty} \frac{\frac{3}{n}-\frac{4}{n^2}}{1-\frac{3}{n}} = 0$, so the Test for Divergence does not apply.

To apply the Integral Test, we need to show that the function $f(x)$ such that $a_n = f(n)$ is continuous, positive, and decreasing on the interval [3,∞).

For this specific series, $f(x) = \frac{3x-4}{x^2-2x}$, $x \geq 3$. This function is the quotient of continuous polynomials $3x-4$ and x^2-2x, and the denominator is nonzero on the domain [3,∞), so $f(x)$ is continuous. On [3,∞), the function is positive since both numerator and denominator are positive. Finally,

$$f(x) = \frac{3x-4}{x^2-2x}$$

so

$$\begin{aligned} f'(x) &= \frac{\left(x^2-2x\right)\cdot 3 - (3x-4)\cdot(2x-2)}{\left(x^2-2x\right)^2} \\ &= \frac{3x^2-6x-6x^2+14x-8}{\left(x^2-2x\right)^2} \\ &= \frac{-3x^2+8x-8}{\left(x^2-2x\right)^2} \\ &< 0 \end{aligned}$$

because the numerator is negative on [3,∞) and the denominator is positive. (Actually, the numerator is always negative, but we only care about [3,∞).) Thus the function is decreasing, and we can apply the Integral Test.

We evaluate

$$\begin{aligned} &\int_3^{\infty} \frac{3x-4}{x^2-2x}\,\mathrm{d}x \\ &= \lim_{t\to\infty} \int_3^{t} \frac{3x-4}{x^2-2x}\,\mathrm{d}x \end{aligned}$$

For this integration, we'll have to use partial fractions. Let's compute the indefinite integral, and then we'll use that antiderivative to finish the improper integral.

We evaluate $\displaystyle\int \frac{3x-4}{x^2-2x}\,\mathrm{d}x$.

The degree of the numerator is 1, and the degree of the denominator is 2. Since the degree of the numerator is less than the degree of the denominator, we are ready to begin the partial fractions process.

$$\frac{3x-4}{x^2-2x} = \frac{3x-4}{x\,(x-2)}$$

[1]Stewart, *Calculus, Early Transcendentals*, p. 726, #20.

x is a distinct linear factor, as is $x-2$. We have

$$\frac{3x-4}{x\left(x-2\right)}=\frac{A}{x}+\frac{B}{x-2}$$

The LCD is $x\left(x-2\right)$, and we multiply both sides of this identity by the LCD to get

$$\begin{aligned}\frac{3x-4}{x\left(x-2\right)}\cdot x\left(x-2\right)&=\frac{A}{x}\cdot x\left(x-2\right)+\frac{B}{x-2}\cdot x\left(x-2\right)\\3x-4&=A\left(x-2\right)+Bx\\3x-4&=Ax-2A+Bx\\3x-4&=Ax+Bx-2A\\3x-4&=\left(A+B\right)x+\left(-2A\right)\end{aligned}$$

Since this equation is an identity, *i.e.*, it is true for all allowable values of x, we equate coefficients to get the system of equations

$$\begin{cases}3&=&A&+&B\\-4&=&-2A\end{cases}$$

The bottom equation gives $A=2$ and substituting into the top equation gives $B=1$. The partial fraction decomposition is

$$\frac{3x-4}{x^2-2x}=\frac{2}{x}+\frac{1}{x-2}$$

so our integral becomes

$$\begin{aligned}\int\frac{3x-4}{x^2-2x}\,\mathrm{d}x&=\int\frac{2}{x}+\frac{1}{x-2}\,\mathrm{d}x\\&=\int\frac{2}{x}\,\mathrm{d}x+\int\frac{1}{x-2}\,\mathrm{d}x\\&=2\ln|x|+\ln|x-2|\end{aligned}$$

and since the domain is $[3,\infty)$, we have

$$=2\ln\left(x\right)+\ln\left(x-2\right)$$

Now

$$\begin{aligned}&\lim_{t\to\infty}\int_3^t\frac{3x-4}{x^2-2x}\,\mathrm{d}x\\&=\lim_{t\to\infty}\left[2\ln\left(x\right)+\ln\left(x-2\right)\right]_3^t\\&=\lim_{t\to\infty}\Big[\left(2\ln\left(t\right)+\ln\left(t-2\right)\right)-\left(2\ln\left(3\right)+\ln\left(3-2\right)\right)\Big]\\&=\infty\end{aligned}$$

So the improper integral is divergent, and thus by the Integral Test, our series $\displaystyle\sum_{n=3}^{\infty}\frac{3n-4}{n^2-2n}$ is divergent.