Determine whether the series converges or diverges.[1]

$$\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2}$$

We suspect convergence because the terms of the series are similar to $\sum \frac{n^2}{n^4} = \sum \frac{1}{n^2}$, which is a convergent p-series with $p = 2$.

To apply the Comparison Test, we need to show that $\frac{n^2+n+1}{n^4+n^2} \le \frac{1}{n^2}$, but a little multiplication gives us $n^2\left(n^2+n+1\right) \le 1\left(n^4+n^2\right)$ or $n^4+n^3+n^2 \le n^4+n^2$, which is FALSE. Thus the direct Comparison Test does not apply to this series.

Let's try the Limit Comparison Test with $a_n = \frac{n^2+n+1}{n^4+n^2}$ and $b_n = \frac{1}{n^2}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{n^2+n+1}{n^4+n^2}}{\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{n^2+n+1}{n^4+n^2} \cdot \frac{n^2}{1} \\
&= \lim_{n\to\infty} \frac{n^2+n+1}{n^2+1} \\
&= \lim_{n\to\infty} \frac{\frac{n^2}{n^2}+\frac{n}{n^2}+\frac{1}{n^2}}{\frac{n^2}{n^2}+\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{1+\frac{n}{n^2}+\frac{1}{n^2}}{1+\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{1+0+0}{1+0} \\
&= 1
\end{aligned}$$

Since the limit is a positive, finite number and $\sum \frac{1}{n^2}$ is a known convergent p-series, our series

$$\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2}$$

is convergent by the Limit Comparison Test.

[1] Stewart, *Calculus, Early Transcendentals*, p. 731, #20.