Determine whether the series converges or diverges.[1]

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Do we suspect convergence? …divergence? Which is larger, $n!$ or n^n? We're not sure.

Certainly, the terms of the series are positive.

Note that

$$\begin{aligned} \frac{n!}{n^n} &= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (n-1) \cdot n}{n \cdot n \cdot n \cdot n \cdot \cdots \cdot n \cdot n} \\ &= \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdot \frac{4}{n} \cdot \cdots \cdot \frac{n-1}{n} \cdot \frac{n}{n} \\ &\le \frac{1}{n} \cdot \frac{2}{n} \cdot 1 \cdot 1 \cdot \cdots \cdot 1 \cdot 1 \\ &= \frac{2}{n^2} \end{aligned}$$

So

$$\frac{n!}{n^n} \le \frac{2}{n^2}$$

which is a known convergent p-series, thus our series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

is convergent by the Comparison Test.

[1]Stewart, *Calculus, Early Transcendentals*, p. 731, #30.