46. *Let $\sum a_n$ be a series with positive terms and let $r_n = a_{n+1}/a_n$. Suppose that $\lim_{n\to\infty} r_n = L < 1$, so $\sum a_n$ converges by the Ratio Test. As usual, we let R_n be the remainder after n terms, that is*[1]

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

(a) *If $\{r_n\}$ is a decreasing sequence and $r_{n+1} < 1$, show, by summing a geometric series, that*

$$R_n \le \frac{a_{n+1}}{1 - r_{n+1}}$$

$$\begin{aligned}
R_n &= a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4} + \cdots \\
&= a_{n+1}\left(1 + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+3}}{a_{n+1}} + \frac{a_{n+4}}{a_{n+1}} + \cdots\right) \\
&= a_{n+1}\left(1 + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+3}}{a_{n+2}} \cdot \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+4}}{a_{n+3}} \cdot \frac{a_{n+3}}{a_{n+3}} \cdot \frac{a_{n+2}}{a_{n+1}} + \cdots\right) \\
&= a_{n+1}\left(1 + r_{n+1} + r_{n+2}r_{n+1} + r_{n+3}r_{n+2}r_{n+1} + \cdots\right)
\end{aligned}$$

and since R_n is decreasing,

$$\begin{aligned}
&\le a_{n+1}\left(1 + r_{n+1} + r_{n+1}^2 + r_{n+1}^3 + \cdots\right) \\
&= \frac{a_{n+1}}{1 - r_{n+1}}
\end{aligned}$$

Thus, if $\{r_n\}$ is a decreasing sequence, then $R_n \le \frac{a_{n+1}}{1-r_{n+1}}$.

(b) *If $\{r_n\}$ is an increasing sequence, show that*

$$R_n \le \frac{a_{n+1}}{1 - L}$$

If $\{r_n\}$ is increasing and $r_n \to L$ as $n \to \infty$, then $r_n < L$ for all n. So

$$\begin{aligned}
R_n &= a_{n+1}\left(1 + r_{n+1} + r_{n+2}r_{n+1} + r_{n+3}r_{n+2}r_{n+1} + \cdots\right) \\
&\le a_{n+1}\left(1 + L + L^2 + L^3 + \cdots\right) \\
&= \frac{a_{n+1}}{1 - L}
\end{aligned}$$

Thus, if $\{r_n\}$ is an increasing sequence, then $R_n \le \frac{a_{n+1}}{1-L}$.

48. *Use the sum of the first 10 terms to approximate the sum of the series*

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

Use the above results to estimate the sum.

$$s_{10} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \frac{6}{64} + \frac{7}{128} + \frac{8}{256} + \frac{9}{512} + \frac{10}{1024} \approx 1.988281$$

We compute the ratios $r_n = \frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} = \frac{1}{2}\left(1 + \frac{1}{n}\right)$, which form a decreasing sequence (the derivative of the corresponding function is $-\frac{1}{2n^2} < 0$. Then

$$r_{11} = \frac{11+1}{2 \cdot 11}$$

[1]Stewart, *Calculus, Early Transcendentals*, p. 743, #46, 48.

$$= \frac{6}{11}$$
$$< 1$$

so the error is

$$\begin{aligned} R_{10} &\le \frac{a_{11}}{1 - r_{11}} \\ &= \frac{\frac{11}{2048}}{1 - \frac{6}{11}} \\ &= \frac{121}{10{,}240} \\ &\approx 0.0118 \end{aligned}$$