Find the radius of convergence and the interval of convergence of the series.[1]

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$$

We apply the Ratio Test to our series

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}x^{n+1}}{(n+1)^2}}{\frac{(-1)^n x^n}{n^2}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}\, x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(-1)^n\, x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{x^{n+1}}{x^n} \cdot \frac{(n+1)^2}{n^2} \right| \\
&= \lim_{n\to\infty} \left| -1 \cdot x \cdot \left(\frac{n+1}{n}\right)^2 \right| \\
&= 1 \cdot |x| \cdot \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 \\
&= 1 \cdot |x| \cdot 1 \\
&= |x|
\end{aligned}$$

The power series is convergent if this limit is less than one, so we solve $|x| < 1 \Rightarrow -1 < x < 1$. Thus $R = 1$. Now we must test the series at the endpoints.

When $x = 1$, our series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n (1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which is a convergent alternating series. (Alternatively, the series $\sum \frac{1}{n^2}$ is a positive-termed, convergent p-series, and is therefore absolutely convergent, so we know the alternating version converges.)

When $x = -1$, our series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(1)^n}{n^2}$ which is a convergent p-series.

Thus, the interval of convergence for our series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$ is [−1,1].

[1] Stewart, *Calculus, Early Transcendentals*, p. 751, #6.