Find the radius of convergence and the interval of convergence of the series.[1]

$$\sum_{n=1}^{\infty} \frac{n^2 x^n}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}$$

Note that

$$\begin{aligned}
a_n &= \frac{n^2 x^n}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)} \\
&= \frac{n^2 x^n}{(2 \cdot 1) \cdot (2 \cdot 2) \cdot (2 \cdot 3) \cdot \cdots \cdot (2 \cdot n)} \\
&= \frac{n^2 x^n}{2^n \cdot 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n)} \\
&= \frac{n^2 x^n}{2^n \cdot n!} \\
&= \frac{n^2 x^n}{2^n \cdot (n-1)! \cdot n} \\
&= \frac{n \cdot x^n}{2^n \cdot (n-1)!}
\end{aligned}$$

Let's apply the Ratio Test to our series.

$$\begin{aligned}
&\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&\quad = \lim_{n\to\infty} \left| a_{n+1} \cdot \frac{1}{a_n} \right| \\
&\quad = \lim_{n\to\infty} \left| \frac{(n+1) \cdot x^{n+1}}{2^{n+1} \cdot ((n+1)-1)!} \cdot \frac{2^n \cdot (n-1)!}{n \cdot x^n} \right| \\
&\quad = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{2^n}{2^{n+1}} \cdot \frac{(n+1)}{n} \cdot \frac{(n-1)!}{n!} \right| \\
&\quad = \lim_{n\to\infty} \left| x \cdot \frac{1}{2} \cdot \left(1 + \frac{1}{n}\right) \cdot \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-2) \cdot (n-1)}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n} \right| \\
&\quad = \left| \frac{x}{2} \right| \cdot \lim_{n\to\infty} \left| \left(1 + \frac{1}{n}\right) \cdot \frac{1}{n} \right| \\
&\quad = \left| \frac{x}{2} \right| \cdot 1 \cdot 0 \\
&\quad = 0
\end{aligned}$$

Thus, the power series

$$\sum_{n=1}^{\infty} \frac{n^2 x^n}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}$$

is convergent by the Ratio Test with $R = \infty$ and the interval of convergence consists of all real numbers, *i.e.*, $I = (-\infty,\infty)$.

[1] Stewart, *Calculus, Early Transcendentals*, p. 751, #24.