Find a power series representation for the function and determine the interval of convergence.[1]

$$f(x) = \frac{x+a}{x^2+a^2}, \quad a > 0$$

The form for the sum of a geometric series with first term 1 and common ratio x is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

where the series will converge for $|x| < 1$, so we will try to write our function in this form.

$$\begin{aligned}
f(x) &= \frac{x+a}{x^2+a^2} \\
&= \frac{x}{x^2+a^2} + \frac{a}{x^2+a^2} \\
&= \frac{x}{a^2-(-x^2)} + \frac{a}{a^2-(-x^2)} \\
&= \frac{x}{a^2\left(1-\left(-\frac{x^2}{a^2}\right)\right)} + \frac{a}{a^2\left(1-\left(-\frac{x^2}{a^2}\right)\right)} \\
&= \frac{x}{a^2} \cdot \frac{1}{1-\left(-\left(\frac{x}{a}\right)^2\right)} + \frac{1}{a} \cdot \frac{1}{1-\left(-\left(\frac{x}{a}\right)^2\right)}
\end{aligned}$$

Each series has first term 1 and common ratio $-\left(\frac{x}{a}\right)^2$. We get

$$\begin{aligned}
&= \frac{x}{a^2} \cdot \sum_{n=0}^{\infty} \left(-\left(\frac{x}{a}\right)^2\right)^n + \frac{1}{a} \cdot \sum_{n=0}^{\infty} \left(-\left(\frac{x}{a}\right)^2\right)^n \\
&= \frac{x}{a^2} \cdot \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{a^{2n}} + \frac{1}{a} \cdot \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{a^{2n}} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{a^{2n+2}} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{a^{2n+1}}
\end{aligned}$$

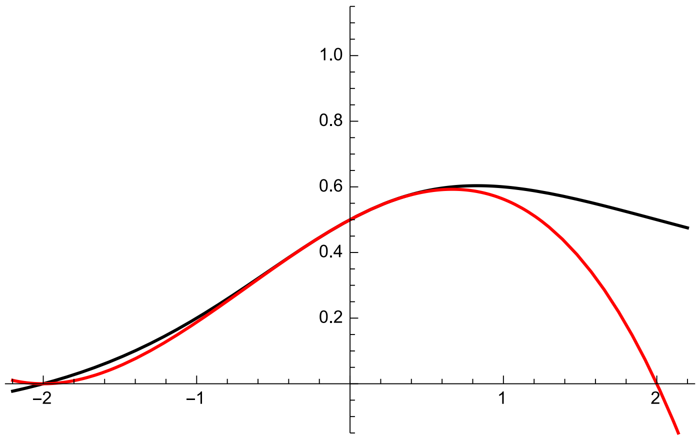
Each of the series converges for

$$\begin{aligned}
\left|-\left(\frac{x}{a}\right)^2\right| &< 1 \\
\left|\frac{x^2}{a^2}\right| &< 1 \\
\left|\frac{x}{a}\right|^2 &< 1 \\
\left|\frac{x}{a}\right| &< 1 \\
-1 < \frac{x}{a} &< 1 \\
-a < x &< a
\end{aligned}$$

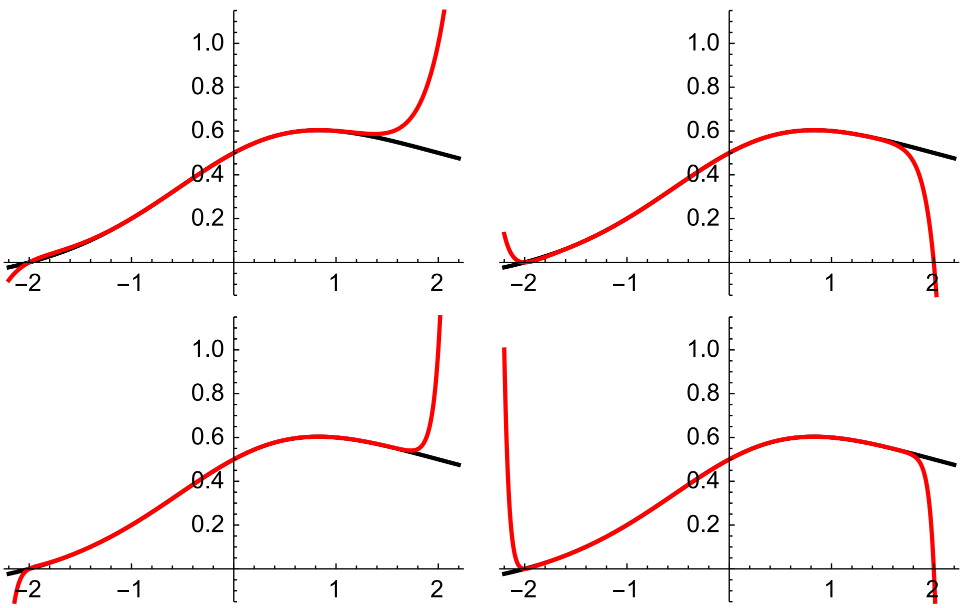
Thus the radius of convergence is $R = a$, and because we obtained the interval of convergence from a geometric form, we have $I = (-a,a)$.

[1] Stewart, *Calculus, Early Transcendentals*, p. 757, #10.

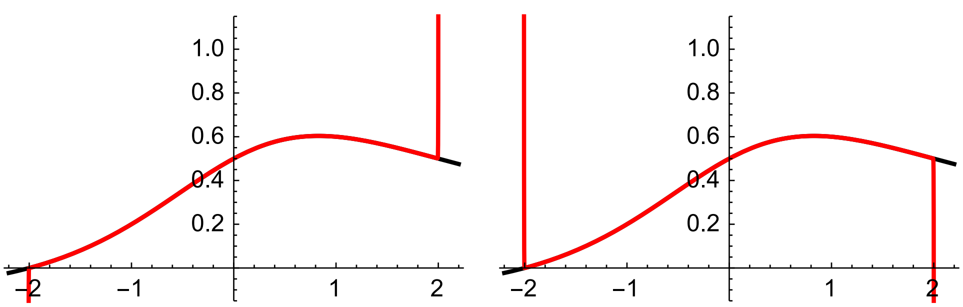
Just for fun, here is the graph of the function $f(x) = \frac{x+2}{x^2+2^2}$ in black along with the second partial sum $\sum_{n=0}^{1} (-1)^n \frac{x^{2n+1}}{2^{2n+2}} + \sum_{n=0}^{1} (-1)^n \frac{x^{2n}}{2^{2n+1}}$ in red.



Similarly, here are the graphs of the 5th, 10th, 15th, and 20th partial sums.



Finally, here are the graphs of the 999th and 1000th partial sum.



We can see that as we add more terms to the partial sums, the series converges to the function on the interval of convergence $(-2,2)$.