Find a power series representation for the function and determine the interval of convergence.[1]

$$f(x) = \left(\frac{x}{2-x}\right)^3$$

Note that if $y = \frac{1}{t}$, then $\frac{dy}{dt} = -\frac{1}{t^2}$ and $\frac{dy^2}{d^2t} = \frac{2}{t^3}$. This suggests that if we were to write a power series for a function of the form $\frac{1}{t}$, then we will be able to obtain a power series for $2 \cdot \frac{1}{t^3}$ by repeated differentiation.

$$\begin{aligned} \frac{1}{2-x} &= \frac{1}{2\left(1-\frac{x}{2}\right)} \\ &= \frac{1}{2} \cdot \frac{1}{1-\frac{x}{2}} \\ &= \frac{1}{2} \cdot \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n \end{aligned}$$

So we get

$$\begin{aligned} \frac{d}{dx}\left[\frac{1}{2-x}\right] &= \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n\right] \\ \frac{1}{(2-x)^2} &= \sum_{n=1}^{\infty} \frac{n}{2^{n+1}} x^{n-1} \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dx}\left[\frac{1}{(2-x)^2}\right] &= \frac{d}{dx}\left[\sum_{n=1}^{\infty} \frac{n}{2^{n+1}} x^{n-1}\right] \\ \frac{2}{(2-x)^3} &= \sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n+1}} x^{n-2} \\ &= \frac{2 \cdot 1}{2^3} x^0 + \frac{3 \cdot 2}{2^4} x^1 + \frac{4 \cdot 3}{2^5} x^2 + \frac{5 \cdot 4}{2^6} x^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2^{n+3}} x^n \end{aligned}$$

Thus,

$$\begin{aligned} f(x) &= \left(\frac{x}{2-x}\right)^3 \\ &= \frac{x^3}{(2-x)^3} \\ &= \frac{x^3}{2} \cdot \frac{2}{(2-x)^3} \\ &= \frac{x^3}{2} \cdot \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2^{n+3}} x^n \end{aligned}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 757, #10.

$$= \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2^{n+4}} x^{n+3}$$

Since this is all based on the geometric series $\frac{1}{2} \cdot \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$ with common ratio $\frac{x}{2}$, we have

$$\left|\frac{x}{2}\right| < 1$$
$$-1 < \frac{x}{2} < 1$$
$$-2 < x < 2$$

So $R = 2$ and the $IOC = (-2{,}2)$.

Here are graphs showing the function in black and the 10th, 20th, 30th, and 40th partial sums in dotted red: