Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.[1]

$$f(x) = x\cos(x)$$

The general form for a Maclaurin series is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Let's organize our work in a table. For this problem, we need enough entries in the table to recognize a pattern for $f^{(n)}(0)$ in terms of n.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$x\cos(x)$	0
1	$-x\sin(x) + \cos(x)$	1
2	$-x\cos(x) - 2\sin(x)$	0
3	$x\sin(x) - 3\cos(x)$	-3
4	$x\cos(x) + 4\sin(x)$	0
5	$-x\sin(x) + 5\cos(x)$	5
6	$-x\cos(x) - 6\sin(x)$	0
7	$x\sin(x) - 7\cos(x)$	-7
8	$x\cos(x) + 8\sin(x)$	0
$\vdots$	$\vdots$	$\vdots$

We got a lot of practice with the product rule for differentiation making that table!

Using the definition of a Maclaurin series and the values in the table, we get

$$\begin{aligned} f(x) &= 0 + \frac{1}{1!}x^1 + 0 - \frac{3}{3!}x^3 + 0 + \frac{5}{5!}x^5 + 0 - \frac{7}{7!}x^7 + 0 + \cdots \\ &= \frac{1}{1!}x^1 - \frac{3}{3!}x^3 + \frac{5}{5!}x^5 - \frac{7}{7!}x^7 + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{(2n+1)!} x^{2n+1} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n+1} \end{aligned}$$

To find the radius and interval of convergence, we use the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1))!}}{\frac{(-1)^n x^{2n+1}}{(2n)!}} \right|$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 771, #16.

$$
\begin{aligned}
&= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}\, x^{2n+3}}{(2n+2)!} \cdot \frac{(2n)!}{(-1)^{n}\, x^{2n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^{n}} \cdot \frac{(2n)!}{(2n+2)!} \cdot \frac{x^{2n+3}}{x^{2n+1}} \right| \\
&= \lim_{n\to\infty} \left| -1 \cdot \frac{1}{(2n+2)\,(2n+1)} \cdot x^2 \right| \\
&= x^2 \cdot \lim_{n\to\infty} \left| \frac{1}{4n^2+6n+2} \right| \\
&= x^2 \cdot 0 \\
&= 0
\end{aligned}
$$

The limit exists and is less than one for all values of x, and thus the series is convergent with $R = \infty$, and $IOC = (-\infty,\infty)$.

Just for fun, we've graphed the function $f(x) = x\cos(x)$ in black and the 10th partial sum of our Maclaurin series in dotted red.