Use the binomial series to expand the function as a power series. State the radius of convergence.[1]

$$f(x) = \sqrt[3]{8+x}$$

The general form for a binomial series is

$$\begin{aligned}(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n \\ &= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots, \quad k \in \mathbb{R}, \quad |x| < 1\end{aligned}$$

We will write the function so it matches the pattern on the left, and then write the power series so it matches the pattern on the right. Then we'll write the power series in a closed form without the binomial coefficients.

$$\begin{aligned}f(x) &= \sqrt[3]{8+x} \\ &= \sqrt[3]{8\left(1+\frac{x}{8}\right)} \\ &= 8^{1/3} \cdot \left(1+\frac{x}{8}\right)^{1/3} \\ &= 2 \cdot \left(1+\frac{x}{8}\right)^{1/3} \\ &= 2 \cdot \sum_{n=0}^{\infty} \binom{1/3}{n} \left(\frac{x}{8}\right)^n \\ &= 2\left[1 + \frac{1}{3}\cdot\left(\frac{x}{8}\right) + \frac{\frac{1}{3}\cdot\frac{-2}{3}}{2\cdot 1}\left(\frac{x}{8}\right)^2 + \frac{\frac{1}{3}\cdot\frac{-2}{3}\cdot\frac{-5}{3}}{3\cdot 2\cdot 1}\left(\frac{x}{8}\right)^3 + \frac{\frac{1}{3}\cdot\frac{-2}{3}\cdot\frac{-5}{3}\cdot\frac{-8}{3}}{4\cdot 3\cdot 2\cdot 1}\left(\frac{x}{8}\right)^4 + \cdots\right]\end{aligned}$$

The pattern really starts with the third term, so we'll hold out the first two terms, and then write the power series in closed form.

$$\begin{aligned}&= 2\left[1 + \frac{1}{3}\cdot\frac{x}{8}\right] + 2\left[-1\cdot\frac{\frac{2}{3^2}}{2!}\cdot\frac{x^2}{8^2} + 1\cdot\frac{\frac{2\cdot 5}{3^3}}{3!}\cdot\frac{x^3}{8^3} + -1\cdot\frac{\frac{2\cdot 5\cdot 8}{3^4}}{4!}\cdot\frac{x^4}{8^4} + \cdots\right] \\ &= 2 + \frac{1}{12}x + 2\left[(-1)^{2-1}\cdot\frac{2}{3^2\cdot 2!}\cdot\frac{x^2}{(2^3)^2} + (-1)^{3-1}\cdot\frac{2\cdot 5}{3^3\cdot 3!}\cdot\frac{x^3}{(2^3)^3} + (-1)^{4-1}\cdot\frac{2\cdot 5\cdot 8}{3^4\cdot 4!}\cdot\frac{x^4}{(2^3)^4} + \cdots\right] \\ &= 2 + \frac{1}{12}x + 2\sum_{n=2}^{\infty}(-1)^{n-1}\frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-4)}{n!\cdot 3^n\cdot 2^{3n}}x^n\end{aligned}$$

To find the radius and interval of convergence, we use the general statement above. The series is convergent on $\left|\frac{x}{8}\right| < 1$ or $-8 < x < 8$, so the radius of convergence is 8. Also, we know that the series is convergent at both endpoints[2] if $k > 0$. Thus the interval of convergence is $[-\infty,\infty]$.

[1] Stewart, *Calculus, Early Transcendentals*, p. 771, #32.
[2] Stewart, *Calculus, Early Transcendentals*, p.767.

Just for fun, we've graphed the function $f(x) = \sqrt[3]{8+x}$ in black and the 10th partial sum of our Taylor series in dotted red.