Use series to approximate the definite integral to within the indicated accuracy.[1]

$$\int_0^{0.5} x^2 e^{-x^2}\,dx, \quad |\text{error}| < 0.001$$

First, we need to write a power series for the integrand, then integrate. When we apply the Fundamental Theorem of Calculus to the antiderivative, we will get a series (*not* a power series) that gives the values of the definite integral. Finally, we can apply an appropriate estimation technique.

We know

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

so

$$\begin{aligned} e^{-x^2} &= \sum_{n=0}^{\infty} \frac{\left(-x^2\right)^n}{n!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \end{aligned}$$

and

$$\begin{aligned} x^2 \cdot e^{-x^2} &= x^2 \cdot \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n!} \end{aligned}$$

Now we integrate to get

$$\begin{aligned} \int_0^{0.5} x^2 e^{-x^2}\,dx &= \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{n!\,(2n+3)}\right]_{x=0}^{x=0.5} \\ &= \left[\sum_{n=0}^{\infty} (-1)^n \frac{(0.5)^{2n+3}}{n!\,(2n+3)}\right] - \left[\sum_{n=0}^{\infty} (-1)^n \frac{0^{2n+3}}{n!\,(2n+3)}\right] \\ &= \left[\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2)^{2n+3} \cdot n!\,(2n+3)}\right] - 0 \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n+3} n!\,(2n+3)} \end{aligned}$$

Note that this series is a number (as all definite integrals are), not a power series. Since the series is alternating, we can apply Alternating Series Estimation. The term with $n = 2$ is $\frac{1}{1792} < 0.001$, so we use

$$\sum_{n=0}^{1} (-1)^n \frac{1}{2^{2n+3} n!\,(2n+3)} = \frac{1}{24} - \frac{1}{160} \approx 0.0354$$

Thus,

$$\int_0^{0.5} x^2 e^{-x^2}\,dx \approx 0.0354$$

with an error of less than 0.001.

[1] Stewart, *Calculus, Early Transcendentals*, p. 772, #60.