Chapter 1

Functions and Models

1.1 Four Ways to Represent a Function

**Definition.** A *function* \( f \) is a rule that assigns to each element \( x \) in a set \( D \) exactly one element, called \( f(x) \), in a set \( E \).

- The set \( D \) is called the *domain* of the function, i.e., the set of all possible \( x \)'s.
- The *range* of \( f \) is the set of all possible values of \( f(x) \) as \( x \) varies throughout the domain.

**Example 1.1.** If \( f(x) = x^2 + 3x + 2 \) and \( h \neq 0 \), evaluate \( \frac{f(a + h) - f(a)}{h} \).
Example 1.2. A rectangle has area $16 \text{ m}^2$. Express the perimeter of the rectangle as a function of the length of one of its sides.

1.1.1 Piecewise Defined Functions

Example 1.3. A function $f$ is defined by

$$f(x) = \begin{cases} 3 - \frac{1}{2}x & \text{if } x \leq 2 \\ 2x - 5 & \text{if } x > 2 \end{cases}$$

Evaluate $f(-2), f(0),$ and $f(2)$ and sketch the graph.
1.1.2 Symmetry

- If a function $f$ satisfies $f(-x) = f(x)$ for every number $x$ in its domain, then $f$ is called an even function.
- If $f$ satisfies $f(-x) = -f(x)$ for every number $x$ in its domain, then $f$ is called an odd function.

Example 1.4. Determine whether each of the following functions is even, odd, or neither odd nor even.

a) $f(x) = x^5 + x$

b) $g(x) = 1 - x^4$

c) $h(x) = 2x - x^2$
1.1.3 Domain

Example 1.5. Find the domain of each function.

a) \( F(x) = x^2 - 2x + 1 \)  \quad b) \( H(t) = 4 - t^2 \)  \quad c) \( g(t) = \frac{7}{\sqrt{2} - t} \)
1.2 Mathematical Models

A mathematical model is a mathematical description—often by means of a function or an equation—of a real-world phenomenon such as size of a population, demand for a product, speed of a falling object, life expectancy of a person at birth, or cost of emission reductions, etc. There are many different types of functions that can be used to model relationships observed in the real world.

1.2.1 Linear Models

When we say that $y$ is a linear function of $x$, we mean that the graph of the function is a line. So, we can use the slope-intercept form of the equation of a line to write a formula for the function as

$$y = f(x) = mx + b;$$

where $m$ is the slope of the line and $b$ is the $y$-intercept.

A characteristic feature of linear functions is that they grow at a constant rate.

**Example 1.6.** The manager of a furniture factory finds that it costs $2200 to manufacture 100 chairs in one day and $4800 to produce 300 chairs in one day.

a) Express the cost as a function of the number of chairs produced, assuming that it is linear. Then sketch a graph.

b) What is the slope of the graph and what does it represent?

c) What is the $y$-intercept and what does it represent?
1.2.2 Power Functions

A function of the form \( f(x) = x^a \), where \( a \) is constant, is called a power function.

Example 1.7. Sketch the graphs of each power function using your calculator or library of functions.

a) \( y = x \)  

b) \( y = x^2 \)  

c) \( y = x^3 \)  

d) \( y = x^4 \)  

e) \( y = x^5 \)  

Example 1.8. Sketch the graphs of each power function using your calculator or library of functions.

a) \( f(x) = \sqrt{x} \)  

b) \( g(x) = \sqrt[3]{x} \)  

c) \( h(x) = \frac{1}{x} \)
1.2.3 **Polynomials**

A function $P$ is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $n$ is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the **coefficients** of the polynomial.

A polynomial is a sum or difference of power functions when $n \geq 0$.

1.2.4 **Rational Functions**

A **rational function** $f$ is a ratio of two polynomials

$$\frac{P(x)}{Q(x)},$$

where $P$ and $Q$ are polynomials, and $Q \neq 0$.

**Example 1.9.** For the function $f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$, graph on your graphing calculator, and sketch it on the grid.

1.2.5 **Algebraic Functions**

A function $f$ is called an **algebraic function** if it can be constructed using algebraic operations (addition, subtraction, multiplication, division, and taking roots) starting with polynomials.
Example 1.10. Graph $y = x\sqrt{x + 3}$, $y = \sqrt{x^2 - 25}$, and $y = x^{(2/3)}(x - 2)^2$ on your calculator and sketch it on the grids.

1.2.6 Trigonometric Functions

In calculus, the convention is that radian measure is always used (except when otherwise indicated).

\[ f(x) = \sin x \quad \text{and} \quad g(x) = \cos x \]

- For both the sine and cosine functions, the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$.
- For all values of $x$ we have $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$
- The sine and cosine functions are periodic functions and have a period $2\pi$.
- The zeros of the sine functions occurs when $x = n\pi$.
- The tangent function is related to the sine and cosine functions by the equation \[ \tan x = \frac{\sin x}{\cos x}. \]
- The remaining three trigonometric functions: cosecant, secant, and cotangent are the reciprocals of the sine, cosine, and tangent functions.
1.2.7 Exponential Functions

The exponential functions are the functions of the form $f(x) = a^x$, where the base $a$ is a positive constant.

Example 1.11. Graph $y = 2^x$ and $y = 0.5^x$ on your calculator and sketch it on the grid.

1.2.8 Logarithmic Functions

The logarithmic functions are the functions of the form $f(x) = \log_a x$, where the base $a$ is a positive constant. The logarithmic functions are the inverse functions of the exponential functions.

Example 1.12. Graph $y = \log x$ and $y = 10^x$ on your calculator and sketch it on the grid. How do the two functions relate?
1.3 New Functions from Old Function

1.3.1 Transformations of Functions

Recall. A general function with transformations looks like \( af(x - h) + k \).

Let \( f(x) \) be a function. Suppose \( c > 0 \). To obtain the graph of

<table>
<thead>
<tr>
<th>( f(x) \pm c )</th>
<th>( f(x \pm c) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Move ( c ) units to up/down, add/subtract ( c ) units to each ( y )-coordinate, and ( f(x) ) has a vertical shift</td>
<td>Move ( c ) units to the left/right, add/subtract ( c ) units from each ( x )-coordinate, and ( f(x) ) has a horizontal shift</td>
</tr>
</tbody>
</table>

Example 1.13. Given the graph \( f(x) = \sqrt{4 - x^2} \), use transformations to graph \( g(x) = \sqrt{4 - x^2} - 3 \) and \( h(x) = \sqrt{4 - (x - 1)^2} \).
Let $f(x)$ be a function. Suppose $c > 1$. To obtain the graph of

<table>
<thead>
<tr>
<th>$cf(x)$</th>
<th>$f(cx)$</th>
<th>$-f(x)$</th>
<th>$f(-x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stretch/Shrink $f(x)$ vertically by a factor of $c$, multiply $y$-coordinates by $c$</td>
<td>Stretch/Shrink $f(x)$ horizontally by a factor of $c$, divide $x$-coordinates by $c$</td>
<td>Reflection about the $x$-axis, multiply $y$-coordinates by a negative</td>
<td>Reflection about the $y$-axis, multiply $x$-coordinates by a negative</td>
</tr>
</tbody>
</table>

**Example 1.14.** Given the graph $f(x) = \sqrt{4 - x^2}$, use transformations to graph $g(x) = -\sqrt{4 - x^2}$, $h(x) = 2\sqrt{4 - x^2}$, and $k(x) = \sqrt{4 - (-x)^2}$.

![Graphs of $f(x)$, $g(x)$, $h(x)$, and $k(x)$](image)

**Example 1.15.** Sketch the graph $f(x) = \cos \frac{1}{2}x$ and $g(x) = 2 - \cos x$.

![Graphs of $f(x)$ and $g(x)$](image)
1.3.2 Combinations of Functions

We say \((f \circ g)(x) = f(g(x))\) as the function \(f\) is composed with \(g\), i.e., we substitute every \(x\) in \(f\) with the function \(g(x)\). This is also known as the composition of \(f\) and \(g\).

Example 1.16. If \(f(x) = \frac{x}{x+1}\), \(g(x) = x^5\), and \(h(x) = 2x - 5\), then find \(f \circ g \circ h\).

Example 1.17. Find functions \(f\) and \(g\) such that \(f \circ g = H(x) = \frac{\tan x}{1 + \tan x}\).
1.4 Graphing Calculators and Computers

Example 1.18. Determine an appropriate viewing rectangle for function $f(x) = \sqrt{0.1x + 20}$ and use it to sketch the graph.

Example 1.19. Determine an appropriate viewing rectangle for function $g(x) = \cos 0.001x$ and use it to sketch the graph.
### 1.5 Exponential Functions

An exponential function is a function of the form

\[ f(x) = a^x, \]

where \( a > 0 \) and \( a \neq 1 \).

Recall the exponential function \( y = e^x \), where \( e \approx 2.71828 \ldots \), an irrational number, was discovered by Euler in 1727.

**Example 1.20.** Graph the function \( y = 1 + 2e^x \) using transformations, if needed. *Do not use a calculator.*

![Graph of \( y = 1 + 2e^x \)](image)

**Example 1.21.** Find the exponential function, \( f(x) = Ca^x \) whose graph is given on the grid below.

![Graph of exponential function](image)
**Example 1.22.** A bacterial culture starts with 500 bacteria and doubles in size every half hour.

a) How many bacteria are there after 3 hours?  
b) How many bacteria are there after $t$ hours?  
c) How many bacteria are there after 40 minutes?  

d) Graph the population function and estimate the time for the population to reach 100,000.
1.6 Inverse Functions and Logarithms

1.6.1 Inverse Functions

**Definition.** A function \( f \) is called a one-to-one function if it never takes on the same value twice. That is, \( f(x_1) \neq f(x_2) \) whenever \( x_1 \neq x_2 \).

**Example 1.23.** Determine whether or not \( f(x) = 2x + 3 \) is one-to-one. Why or why not?

**Horizontal Line Test**

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

**Example 1.24.** Determine whether or not \( g(x) = x^2 \) is one-to-one. Why or why not?
**Definition.** Let $f$ be a one-to-one function with domain $A$ and range $B$. Then, its *inverse* function $f^{-1}$ has domain $B$ and range $A$ and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any $y$ in $B$.

**Example 1.25.** Draw the inverse function.

![Graph of a function and its inverse](image)

**Note.** The graph of $f^{-1}$ is obtained by reflecting the graph of $f$ about the line $y = x$.

**Steps to Algebraically Finding the Inverse:**

**Step 1.** Replace $f(x)$ with $y$.

**Step 2.** Switch $x$ and $y$.

**Step 3.** Solve for $y$.

**Step 4.** Replace $y$ with $f^{-1}(x)$.

**Step 5.** Check your work with $f(f^{-1}(x)) = x$ or $f^{-1}(f(x)) = x$. 
Example 1.26. Find the inverse of \( f(x) = x^3 - 2 \).

1.6.2 Logarithmic Functions

**Definition.** The logarithmic function is denoted by

\[ y = \log_a x \]

which is equivalent to \( x = a^y \),

where \( a > 0 \) and \( a \neq 1 \). The value \( a \) is the base, \( y \) is the exponent, and \( x \) is the value.

The equation \( y = \log_a x \) is called the **logarithm form** and \( x = a^y \) is called the **exponential form**.

**Note.** Recall Euler’s constant \( e \). If \( e^y = x \), then \( \log_e x = \ln x = y \) and vice versa. We use the natural logarithm, “\( \ln \)” in place of “\( \log_e \)”.

Example 1.27. Solve the equation \( e^{6-5x} = 10 \).
Laws of Logarithms

Law 1. \( \log_a (xy) = \log_a x + \log_a y \)

Law 2. \( \log_a \left( \frac{x}{y} \right) = \log_a x - \log_a y \)

Law 3. \( \log_a x^r = r \log_a x \)

Example 1.28. Use the laws of logarithms to write \( \ln(a+b) + \ln(a-b) - 2 \ln c \) as a single logarithm.

Change of Base Formula

If \( a \neq 1 \) and \( x \) are positive real numbers, then

\[
\log_a x = \frac{\log x}{\log a} \quad \text{or} \quad \log_a x = \frac{\ln x}{\ln a}
\]

Example 1.29. Solve \( 3^{x-2} = 7 \).
1.6.3 Inverse Trigonometric Functions

<table>
<thead>
<tr>
<th>Inverse Trigonometric Functions</th>
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<tbody>
<tr>
<td>Below are the inverse trigonometric functions with their restricted</td>
</tr>
<tr>
<td>domain:</td>
</tr>
<tr>
<td>1. $\sin^{-1} x = y \iff \sin y = x$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$</td>
</tr>
<tr>
<td>2. $\cos^{-1} x = y \iff \cos y = x$ for $0 \leq y \leq \pi$</td>
</tr>
<tr>
<td>3. $\tan^{-1} x = y \iff \tan y = x$ for $-\frac{\pi}{2} &lt; y &lt; \frac{\pi}{2}$</td>
</tr>
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</table>

Example 1.30. Find the exact value for $\cos^{-1}\left(-\frac{1}{2}\right)$.

Example 1.31. Simplify the expression $\tan(\sin^{-1} x)$. 
Chapter 2

Limits and Derivatives

2.1 The Tangent and Velocity Problems

The word tangent is derived from the Latin word tangens, which means ‘touching’. Thus, a tangent to a curve is a line that touches the curve. In other words, a tangent line should have the same direction as the curve at the point of contact.

Example 2.1. Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

a) Take a point close to $(1, 1)$ on the parabola, $Q(x, x^2)$, and find the slope, $m_{PQ}$, of the secant line $PQ$.

b) As we take the secant line $PQ$ to be smaller and smaller, what does the slope of the tangent line appear to be?

<table>
<thead>
<tr>
<th>$x$</th>
<th>$m_{PQ}$</th>
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<table>
<thead>
<tr>
<th>$x$</th>
<th>$m_{PQ}$</th>
</tr>
</thead>
</table>
c) Determine \( \lim_{Q \to P} m_{PQ} \) and then \( \lim_{x \to 1} m_{PQ} \).

d) Now that you found \( m \) for the tangent line at \( P(1, 1) \), find the equation of the tangent line at \( P(1, 1) \).
Example 2.2. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height in meters $t$ seconds later is given by $y = 10t - 1.86t^2$.

a) Find the average velocity for intervals: $[1, 2], [1, 1.5], [1, 1.1], [1, 1.01], [1, 1.001]$.

b) Estimate the instantaneous velocity when $t = 1$, i.e., find $\lim_{{t \to 1}} m_{PQ}$. 

2.2 The Limit of a Function

**Definition.** We write

\[ \lim_{x \to a} f(x) = L \]

and say “the limit of \( f(x) \), as \( x \) approaches \( a \), equals \( L \)” if we can make the values of \( f(x) \) arbitrarily close to \( L \) (as close to \( L \) as we like) by taking \( x \) to be sufficiently close to \( a \) (on either side of \( a \), but not equal to \( a \)).

**Example 2.3.** Estimate the value for \( \lim_{x \to 1} \frac{x - 1}{x^2 - 1} \).

**Example 2.4.** Estimate the value for \( \lim_{x \to 0} \frac{\sin x}{x} \).
2.2. THE LIMIT OF A FUNCTION

Example 2.5. The Heaviside function \( H \) is defined by

\[
H(t) = \begin{cases} 
0 & \text{if } t < 0 \\
1 & \text{if } t \geq 0 
\end{cases}
\]

What is \( \lim_{t \to 0} H(t) \)?

2.2.1 One-Sided Limits

**Definition.** We write \( \lim_{x \to a^-} f(x) \) and say the **left-hand limit** of \( f(x) \) as \( x \) approaches \( a \), or the limit of \( f(x) \) as \( x \) approaches \( a \) from the left, is equal to \( L \) if we can make the values of \( f(x) \) arbitrarily close to \( L \) by taking \( x \) to be sufficiently close to \( a \) and \( x < a \).

Similarly, we write \( \lim_{x \to a^+} f(x) \) and say the **right-hand limit** of \( f(x) \) as \( x \) approaches \( a \), or the limit of \( f(x) \) as \( x \) approaches \( a \) from the right, is equal to \( L \) if we can make the values of \( f(x) \) arbitrarily close to \( L \) by taking \( x \) to be sufficiently close to \( a \) and \( x > a \).

Example 2.6. Looking at the graph of \( f(x) \), determine \( \lim_{x \to 1^-} f(x) \), \( \lim_{x \to 1^+} f(x) \), and \( \lim_{x \to 1} f(x) \).

Example 2.7. Looking at the graph of \( f(x) \), determine \( \lim_{x \to 3^-} f(x) \), \( \lim_{x \to 3^+} f(x) \), and \( \lim_{x \to 3} f(x) \).
By the definition above, we can see that

\[
\lim_{x \to a} f(x) = L \text{ if only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L
\]

2.2.2 Infinite Limits

Example 2.8. Find \( \lim_{x \to -3^-} \frac{x+2}{x+3} \), \( \lim_{x \to -3^+} \frac{x+2}{x+3} \), \( \lim_{x \to -3} \frac{x+2}{x+3} \).

\[\text{Note. In general, if } \lim_{x \to a^-} f(x) = \pm \infty \text{ and } \lim_{x \to a^+} f(x) = \mp \infty, \text{ then } x = a \text{ is a vertical asymptote.}\]
2.3 Calculating Limits Using the Limit Laws

Limit Laws
Suppose that \( c \) is a constant and the limits

\[
\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)
\]

exist. Then

**Law 1.** \( \lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) \)

**Law 2.** \( \lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x) \)

**Law 3.** \( \lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) \)

**Law 4.** \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if} \quad \lim_{x \to a} g(x) \neq 0 \)

**Example 2.9.** Using the Limit Laws and the graphs of \( f \) and \( g \) to evaluate the following limits, if they exist.

a) \( \lim_{x \to -1} [f(x) + 2g(x)] \)  

b) \( \lim_{x \to 1} [f(x)g(x)] \)  

c) \( \lim_{x \to 0} \frac{f(x)}{g(x)} \)
**More Limit Laws**

Suppose that \( c \) is a constant and the limits

\[
\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)
\]

exist. Then

**Law 5.** \( \lim_{x \to a} [f(x)]^n = \left( \lim_{x \to a} f(x) \right)^n \)

**Law 6.** \( \lim_{x \to a} c = c \)

**Law 7.** \( \lim_{x \to a} x = a \)

**Law 8.** \( \lim_{x \to a} x^n = a^n \), where \( n \) is a positive integer.

**Law 9.** \( \lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} \), where \( n \) is a positive integer. If \( n \) is even, we assume \( a > 0 \).

**Law 10.** \( \lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} \), where \( n \) is a positive integer. If \( n \) is even, we assume \( \lim_{x \to a} f(x) > 0 \).

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**Example 2.10.** Evaluate the limit and justify each step by indicating the appropriate Limit Laws.

a) \( \lim_{x \to 8} (1 + \sqrt[3]{x})(2 - 6x^2 + x^3) \)

b) \( \lim_{x \to 2} \frac{2x^2 + 1}{x^2 + 6x - 4} \)
Direct Substitution Property If \( f \) is a polynomial or a rational function and \( a \) is in the domain of \( f \), then
\[
\lim_{x \to a} f(x) = f(a)
\]

Example 2.11. Evaluate the limit, if it exists.

a) \( \lim_{x \to 4} \frac{x^2 + 5x + 4}{x^2 + 3x - 4} \)

b) \( \lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} \)
Example 2.12. Prove that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.
Example 2.13. Let \( f(x) = \begin{cases} 4 - x^2 & \text{if } x \leq 2 \\ x - 1 & \text{if } x > 2 \end{cases} \)

a) Sketch a graph of \( f(x) \).

b) Find the \( \lim_{x \to 2^-} f(x) \) and \( \lim_{x \to 2^+} f(x) \).

c) Does \( \lim_{x \to 2} f(x) \) exist? Why or why not?

\[ \begin{array}{c}
\hline
\text{The Squeeze Theorem} \\
\text{If } f(x) \leq g(x) \leq h(x) \text{ when } x \text{ is near } a \text{ (except possibly at } a) \text{ and } \\
\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \\
\text{then } \\
\lim_{x \to a} g(x) = L \\
\hline
\end{array} \]

Example 2.14. Prove that the \( \lim_{x \to 0} x^4 \cos \frac{2}{x} = 0 \).
2.4 The Precise Definition of a Limit

Recall in Section 2.2 we use the phrases “x is close to a” and “f(x) gets closer to L” vaguely. In order to be more precise, we discuss the precise definition of a limit.

Example 2.15. For \( f(x) = 3x - 2 \), we can see when \( x \) is close to 2, then \( f(x) \) is close to 4. To be more precise we ask the question, “How close to 2 does \( x \) have to be so that \( f(x) \) differs from 4 by less than 0.1?”

\begin{enumerate}
\item a) Graph \( f(x) \).
\item b) Find a number \( \delta \) such that \( |f(x) - 4| < 0.1 \) if \( 0 < |x - 2| < \delta \).
\item c) Interpret what this \( \delta \) means in context of \( f(x) \).
\item d) Now, let’s find a number \( \delta \) such that \( |f(x) - 4| < 0.001 \) if \( 0 < |x - 2| < \delta \).
\end{enumerate}
Note. The numbers 0.1 and 0.001 are called error tolerances and for 4 to be the precise limit of \( f(x) \) as \( x \) approaches 2, we must not only bring the distance between \( f(x) \) and 4 to be below these error tolerances, we must be able to bring it below any positive number. We write \( \varepsilon \), pronounced epsilon, for this arbitrary positive number.

**Example 2.16.** Find a number \( \delta \) such that \( |f(x) - 4| < \varepsilon \) if \( 0 < |x - 2| < \delta \).

**Definition.** Let \( f \) be a function defined on some open interval containing \( a \) (except possibly at \( a \)) and let \( L \) be a real number. We say that the limit of \( f(x) \) as \( x \) approaches \( a \) is \( L \), and write

\[
\lim_{x \to a} f(x) = L
\]

if for every number \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
\text{if } 0 < |x - a| < \delta \text{ then } |f(x) - L| < \varepsilon
\]
Example 2.17. Prove that \( \lim_{x \to 1} (6 - 7x) = -1 \).

**Prep Work:** Find \( \delta \).  

**Proof:** Prove \( \lim_{x \to 1} (6 - 7x) = -1 \).

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Example 2.18. Prove that \( \lim_{x \to -2} \left( \frac{1}{2}x + 5 \right) = 4 \).

**Prep Work:** Find \( \delta \).  

**Proof:** Prove \( \lim_{x \to -2} \left( \frac{1}{2}x + 5 \right) = 4 \).
### Definitions of Left and Right-hand Limits

**Definition.**

\[ \lim_{x \to a^-} f(x) = L \]

if for every number \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
\text{if } a - \delta < x < a \text{ then } |f(x) - L| < \varepsilon
\]

**Definition.**

\[ \lim_{x \to a^+} f(x) = L \]

if for every number \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
\text{if } a < x < a + \delta \text{ then } |f(x) - L| < \varepsilon
\]

---

**Example 2.19.** Prove that \( \lim_{x \to 0^+} \sqrt{x} = 0 \).

**Prep Work:** Find \( \delta \).  

**Proof:** Prove \( \lim_{x \to 0^+} \sqrt{x} = 0 \).

---

**Example 2.20.** Prove that \( \lim_{x \to a} c = c \). (You are proving Limit Law 7.)
**Example 2.21.** Prove that $\lim_{x \to a} x = a$. (You are proving Limit Law 8.)

**Prep Work:** Find $\delta$.

**Proof:** Prove $\lim_{x \to a} x = a$.

---

**Example 2.22.** Prove that $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$. (You are proving Limit Law 1.)

**Prep Work:** Find $\delta$.

**Proof:** Prove $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.
2.5 Continuity

**Definition.** A function $f$ is *continuous at a number* $a$ if

$$\lim_{x \to a} f(x) = f(a)$$

Notice that this definition implicitly requires three things if $f$ is continuous at $a$:

1. $f(a)$ is defined (that is, $a$ is in the domain of $f$)
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$

**Example 2.23.** Looking at the graph of $f$, at which numbers is $f$ not continuous.

**Note.** If $f$ is defined near $a$, i.e., $f$ is defined on an open interval containing $a$, except perhaps at $a$, we say that $f$ is **discontinuous at $a$** (or $f$ has a discontinuity at $a$) if $f$ is not continuous at $a$.

**Example 2.24.** Use the definition of continuity and the properties of limits to show that $f(x) = x^2 + \sqrt{7-x}$ is continuous at $a = 4$. 
**Definition.** A function \( f \) is *continuous from the right* at a number \( a \) if

\[
\lim_{x \to a^+} f(x) = f(a)
\]

and a function \( f \) is *continuous from the left* at a number \( a \) if

\[
\lim_{x \to a^-} f(x) = f(a)
\]

A function \( f \) is *continuous on an interval* if it is continuous at every number in the interval. If \( f \) is defined only on one side of an endpoint of the interval, we understand ‘continuous at the endpoint’ to mean ‘continuous from the right’ or ‘continuous from the left’.

**Example 2.25.** Use the definition of continuity and the properties of limits to show that \( g(x) = 2 \sqrt{3 - x} \) is continuous on \((-\infty, 3]\).

---

**Theorem 2.5.1.** If \( f \) and \( g \) are continuous at \( a \), and \( c \) is a constant, then the following functions are also continuous at \( a \):

1. \( f \pm g \)
2. \( cf \)
3. \( fg \)
4. \( \frac{f}{g} \) if \( g(a) \neq 0 \)
Example 2.26. Prove $f + g$ is continuous at $a$.

Theorem 2.5.2.

- Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, +\infty)$.
- Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.
Example 2.27. Explain why \( h(x) = \frac{\sin x}{x + 1} \) is continuous at every number in its domain. State the domain.

The following types of functions are continuous at every number in their domains:

- Polynomials
- Rational functions
- Root functions
- Trigonometric functions
- Inverse trigonometric functions
- Exponential functions
- Logarithmic functions

Example 2.28. Find the numbers at which

\[
f(x) = \begin{cases} 
1 + x & \text{if } x \leq 1 \\
\frac{1}{x} & \text{if } 1 < x < 3 \\
\sqrt{x - 3} & \text{if } x \geq 3 
\end{cases}
\]

is discontinuous. At which of these numbers is \( f \) continuous from the right, from the left, or neither? Sketch the graph of \( f \).
The **Intermediate Value Theorem**

Suppose that $f$ is continuous on the closed interval $[a, b]$ and let $N$ be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then, there exists a number $c$ in $(a, b)$ such that $f(c) = N$.

The theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$.

---

**Example 2.29.** Use the Intermediate Value Theorem to show that there is a root of $\sqrt{x} = 1 - x$ in interval $(0, 1)$.


2.6 Limits at Infinity

Example 2.30. Let’s begin by investigating the behavior of the function $f$ defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

as $x$ becomes large, i.e., as $x \to +\infty$.

**a)** Start by sketching the graph.

**b)** What is happening as $x \to +\infty$?

**c)** What is happening as $x \to -\infty$?

---

**Definition.** Let $f$ be a function defined on some interval $(a, +\infty)$. Then

$$\lim_{x \to +\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to $L$ by taking $x$ sufficiently large.

Let $f$ be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to $L$ by taking $x$ sufficiently large negative.
2.6. LIMITS AT INFINITY

**Example 2.31.** Identify any horizontal asymptotes for \( f(x) = \frac{x^2 - 1}{x^2 + 1} \).

**Definition.** The line \( y = L \) is called a **horizontal asymptote** of the curve \( y = f(x) \) if either
\[
\lim_{x \to \pm \infty} f(x) = L \quad \text{or} \quad \lim_{x \to \pm \infty} f(x) = L
\]

**Note.** There is a **vertical asymptote** if
\[
\lim_{x \to a} f(x) = \pm \infty \quad \text{or} \quad \lim_{x \to a^\pm} f(x) = \pm \infty
\]

**Example 2.32.** Sketch the graph of an example of a function \( f \) that satisfies all of the given conditions.

\[
\lim_{x \to 0^+} f(x) = +\infty, \quad \lim_{x \to 0^-} f(x) = -\infty, \quad \lim_{x \to +\infty} f(x) = 1 \quad \lim_{x \to -\infty} f(x) = 1
\]

List all horizontal and vertical asymptotes.
Example 2.33. Find the \( \lim_{x \to +\infty} \frac{1}{x} \) and \( \lim_{x \to -\infty} \frac{1}{x} \).

Theorem 2.6.1. If \( r > 0 \) is a rational number, then \( \lim_{x \to +\infty} \frac{1}{x^r} = 0 \). If \( r > 0 \) is a rational number such that \( x^r \) is defined for all \( x \), then \( \lim_{x \to -\infty} \frac{1}{x^r} = 0 \).

Example 2.34. Find \( \lim_{x \to -\infty} \frac{1 - x - x^2}{2x^2 - 7} \). Example 2.35. Find \( \lim_{x \to -\infty} \left( x + \sqrt{x^2 + 2x} \right) \).
2.6. LIMITS AT INFINITY

Example 2.36. Find \( \lim_{x \to \infty} \frac{x + 2}{\sqrt{9x^2 + 1}}. \)

2.6.1 Infinite Limits at Infinity

Example 2.37. Find \( \lim_{x \to -\infty} x^4 + x^5. \)

Example 2.38. Find \( \lim_{x \to \infty} \frac{x + x^3 + x^5}{1 - x^2 + x^4}. \)

Example 2.39. Find \( \lim_{x \to (\pi/2)^+} e^{\tan x}. \)
Example 2.40. Find the limits as $x \to +\infty$ and $x \to -\infty$. Use this information, together with intercepts, to give a rough sketch of the graph.

$$y = x^3(x + 2)^2(x - 1)$$

Example 2.41. Use the Squeeze Theorem to evaluate $\lim_{x \to \infty} \frac{\sin x}{x}$. 
2.7 Derivatives and Rates of Change

2.7.1 Tangents

Recall, Section 2.1, when we found the slope to a tangent line.

**Example 2.42.** Find the slope of the tangent line for \( y = f(x) \) at point \( P(a, f(a)) \).

**Definition.** The tangent line to the curve \( y = f(x) \) at the point \( P(a, f(a)) \) is the line through \( P \) with slope

\[
m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]

provided that this limit exists.

**Note.** Allowing \( h = x - a \), then we substitute \( x = a + h \) so that we obtain another formula:

\[
m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

**Example 2.43.** Let’s revisit the first example in Section 2.1 and confirm our results. Find an equation of a tangent line to the parabola \( y = x^2 \) at the point \( (1, 1) \).
Example 2.44. Find an equation of the tangent line to the curve \( y = \frac{2x}{(x + 1)^2} \) at the point \((0, 0)\).

2.7.2 Velocities

Recall. In Section 2.1 we poked around with average velocity and instantaneous velocity using the same ideas as finding a tangent line.

Example 2.45. Find the average velocity for \( y = f(x) \), the position function, from time \( t = a \) to \( t = a + h \).

In general, average velocity is

\[
m_{PQ} = \frac{f(a + h) - f(a)}{h}
\]

and the velocity or \textit{instantaneous velocity} at \( t = a \) is

\[
v(a) = \lim_{{h \to 0}} \frac{f(a + h) - f(a)}{h}
\]
Example 2.46. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height (in meters) after $t$ seconds is given by $H = 10t - 1.86t^2$.

a) Find the velocity of the rock after one second.

b) Find the velocity of the rock when $t = a$.

c) When will the rock hit the surface.

d) With what velocity will the rock hit the surface?

2.7.3 Derivatives

**Definition.** The derivative of a function $f$ at a number $a$, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.

**Note.** Allowing $x = a + h$, then we substitute $h = x - a$ so that we obtain another formula:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
Example 2.47. Find the derivative of the function \( f(x) = x^2 - 7x + 6 \) at the number \( a \) using the definition, i.e., find \( f'(a) \).

Note. We call \( f'(a) \) the slope of the tangent line to \( y = f(x) \) at point \( P(a, f(a)) \), i.e., \( f'(a) \) will give the slope of the tangent line at any \( x \) value.

Example 2.48. Find the equation of the tangent line to \( f(x) = x^2 - 7x + 6 \) at \( (2, -4) \). Verify your results by graphing both in your calculator and zooming in at the point \( (2, -4) \).
2.7.4 Rates of Change

Recall. The average rate of change is given by \( \frac{\Delta y}{\Delta x} \) for points \( P(x_1, y_1) \) and \( Q(x_2, y_2) \). Furthermore, we call this the slope of the secant line \( PQ \).

In general, average rate of change is

\[
m_{PQ} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]

and the **instantaneous rate of change** at \( x = x_1 \) is

\[
\lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]

Hence, this is the derivative \( f'(x_1) \). In general, the derivative \( f'(a) \) is the instantaneous rate of change of \( y = f(x) \) with respect to \( x \) when \( x = a \).
Example 2.49. The cost (in dollars) of producing \( x \) units of a certain commodity is
\[
C(x) = 5000 + 10x + 0.05x^2.
\]

\textbf{a)} What is the meaning of the derivative \( C'(x) \)? What are its units?

\textbf{b)} Find the instantaneous rate of change of \( C \) with respect to \( x \) when \( x = 100 \).

\textbf{c)} What does \( C'(100) \) mean?
2.8 The Derivative as a Function

Before we considered the derivative of a function at a fixed number $a$. We now let $a$ vary and replace $a$ with $x$ such that the derivative of $f$ for any $x$ in which the limit exists is given as

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Note, geometrically, $f'(x)$ is still interpreted as the slope of the tangent line to the graph $f$ at point $(x, f(x))$.

**Example 2.50.** If $f(x) = x^4 + 2x$, find $f'(x)$. Graph both and compare.
Example 2.51. Find the derivative of $f(x) = x + \sqrt{x}$. State the domain of $f$ and its derivative.

**Definition.** A function $f$ is *differentiable at* $a$ if $f'(a)$ exists. It is *differentiable on an open interval* $(a, b)$ (or $(a, +\infty)$ or $(-\infty, a)$ or $(-\infty, +\infty)$) if it is differentiable at every number in the interval.

Example 2.52. Where is the function $f(x) = |x|$ differentiable?
Theorem 2.8.1. If $f$ is differentiable at $a$, then $f$ is continuous at $a$.

\textbf{Note.} \emph{WARNING}: The converse of this theorem is not true. There are functions that are continuous, but not differentiable.

\textbf{Example 2.53.} Prove Theorem 2.8.1.
2.8.1 Functions that Fail to be Differentiable

As we saw in an earlier example, \( f(x) = |x| \) was not differentiable at zero since the limit did not exist at zero. Here are some more examples of when this occurs:

\[
\begin{align*}
 f(x) & \quad f(x) & \quad f(x) \\
 \end{align*}
\]

2.8.2 Higher Derivatives

If \( f \) is a differentiable function, then \( f' \) is also a function and may have a derivative of its own, denoted by \( (f')' = f'' \) and is called the second derivative. We could go onto to say if \( f' \) is a differentiable function, then \( f'' \) is also a function and may have a derivative of its own, denoted by \( ((f')')' = f''' \) and is called the third derivative, etc. for higher derivatives.
Example 2.54. Given the graph of $f, f'$, and $f''$. Identify each curve.
Example 2.55. If \( f(x) = 2x^2 - x^3 \), find \( f', f'', \) and \( f''' \). Sketch all four functions on the same grid and label.
Chapter 3

Differentiation Rules

3.1 Derivatives of Polynomials & Exponential Functions

Other Notation
Leibniz notation is the notation Gottfried Leibniz used while discovering calculus. We write traditionally $y = f(x)$, where $f'(x)$ is the derivative. In Leibniz notation we write

$$f'(x) = y' = \frac{dy}{dx}$$

If we want to write the derivative at a number $a$, we write

$$\frac{dy}{dx} \bigg|_{x=a} = f'(a)$$

We say $\frac{dy}{dx}$ as “the derivative with respect to $x$.”
3.1.1 Power Functions

Example 3.1. Find the derivative of $y = x^3$, $y = x^2$, $y = x$, and $y = c$ by using the definition of the derivative.

\[ \text{Power Rule} \]
If $n$ is any real number, then
\[ \frac{d}{dx}(x^n) = nx^{n-1} \]

Example 3.2. Differentiate using the Power Rule.

a) $F(x) = x^8$

b) $g(t) = \frac{1}{t^5}$

c) $f(x) = \sqrt[3]{x^3}$
3.1. DERIVATIVES OF POLYNOMIALS & EXPONENTIAL FUNCTIONS

3.1.2 New Derivatives from Old

**Constant Multiple Rule**

If $c$ is a constant and $f$ is a differentiable function, then

$$\frac{d}{dx}(cf(x)) = c \frac{d}{dx}f(x)$$

**Example 3.3.** Differentiate.

a) $F(x) = \frac{3}{7}x^{11}$  

b) $g(t) = \frac{2}{t^3}$  

c) $f(x) = -\sqrt[3]{x^8}$

**Sum & Difference Rule**

If $f$ and $g$ are both differentiable, then

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

**Example 3.4.** Prove $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$. 
Example 3.5. Differentiate.

a) \( h(x) = (x - 2)(2x + 3) \)  

b) \( g(t) = \sqrt{t} - \frac{1}{\sqrt{t}} \)  

c) \( f(x) = \frac{x^2 - 2\sqrt{x}}{x} \)

3.1.3 Exponential Functions

Example 3.6. If \( y = e^x \), what is \( \frac{dy}{dx} \)?

Note. In general, \( \frac{d}{dx}(e^x) = e^x \).
Example 3.7. Find an equation of a tangent line to $f(x) = x^4 + 2e^x$ at point (0,2).
3.2 Product and Quotient Rules

<table>
<thead>
<tr>
<th>Product Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>If $f$ and $g$ are both differentiable, then</td>
</tr>
<tr>
<td>$\frac{d}{dx}[f(x)g(x)] = f(x) \cdot \frac{d}{dx}g(x) + \frac{d}{dx}f(x) \cdot g(x)$</td>
</tr>
</tbody>
</table>

**Example 3.8.** Prove the Product Rule.

**Example 3.9.** Differentiate.

a) $f(x) = \sqrt{x}e^x$

b) $z = w^{3/2}(w + ce^w)$
**Quotient Rule**

If $f$ and $g$ are both differentiable, then

$$
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{d}{dx} f(x) \cdot g(x) - f(x) \cdot \frac{d}{dx} g(x)}{[g(x)]^2},
$$

where $g(x) \neq 0$

*Example 3.10. Differentiate.*

**a)** $y = \frac{t - \sqrt{t}}{t^{1/3}}$

**b)** $f(x) = \frac{1 - xe^x}{x + e^x}$
Example 3.11. Find an equation of the tangent line to $y = \frac{e^x}{x}$ at $(1, e)$.

\[
\begin{array}{l}
\frac{d}{dx}(c) = 0 \\
\frac{d}{dx}(e^x) = e^x \\
\frac{d}{dx}(x^n) = nx^{n-1} \\
(cf)' = cf' \\
(f \pm g)' = f' \pm g' \\
(fg)' = fg' + f'g \\
\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}
\end{array}
\]
3.3 Derivatives of Trigonometric Functions

<table>
<thead>
<tr>
<th>Trigonometric Identities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin^2 x + \cos^2 x = 1$</td>
</tr>
<tr>
<td>$1 + \tan^2 x = \sec^2 x$</td>
</tr>
<tr>
<td>$\sin 2x = 2 \sin x \cos x$</td>
</tr>
</tbody>
</table>

**Example 3.12.** Using the definition of a derivative, prove that if $f(x) = \cos x$, then $f'(x) = -\sin x$. Graph $f$ and $f'$ to verify your results.
**Example 3.13.** Using the Quotient Rule, prove that if $g(x) = \tan x$, then $g'(x) = \sec^2 x$.

**Example 3.14.** Prove that if $h(x) = \sec x$, then $h'(x) = \sec x \tan x$. 
Example 3.15. Differentiate.

a) \( f(x) = \sqrt{x} \sin x \)

b) \( y = \frac{1 + \sin x}{x + \cos x} \)

c) \( y = \csc \theta (\theta + \cot \theta) \)

d) \( g(t) = 4 \sec t + \tan t \)
Example 3.16. Find an equation of the tangent line to \( y = \sec x - 2 \cos x \) at \((\pi/3, 1)\). Verify your results by sketching \( y \) and the tangent line.
3.3. DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

Example 3.17. Find the limit.

a) \[ \lim_{\theta \to 0} \frac{\cos \theta - 1}{\sin \theta} \]

b) \[ \lim_{t \to 0} \frac{\sin^2 3t}{t^2} \]

\[ \frac{d}{dx} (\sin x) = \cos x \quad \frac{d}{dx} (\tan x) = \sec^2 x \quad \frac{d}{dx} (\sec x) = \sec x \tan x \]

\[ \frac{d}{dx} (\cos x) = -\sin x \quad \frac{d}{dx} (\csc x) = -\csc x \cot x \quad \frac{d}{dx} (\cot x) = -\csc^2 x \]
3.4 The Chain Rule

The Chain Rule

If $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at $x$ and $F'$ is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 3.18. Prove the Chain Rule.
Example 3.19. Write the function of the form $f(g(x))$, i.e., identify the inner function $u = g(x)$ and the outer function $y = f(u)$, and find the derivative.

a) $y = \sqrt{4 + 3x}$

b) $y = \tan(\sin x)$
Example 3.20. Find the derivative of the function.

a) \( f(x) = (1 + x^4)^{2/3} \)  

b) \( f(t) = \sqrt{1 + \tan t} \)

c) \( y = e^{-5x} \cos 3x \)  

d) \( y = \sin (\sin (\sin x)) \)
e) \( y = e^{k \tan x} \), where \( k \) is a constant

f) \( G(y) = \left( \frac{y^2}{y + 1} \right)^5 \)

Example 3.21. Find an equation of the tangent line to \( y = x^2 e^{-x} \) at \((1, 1/e)\).
Example 3.22. If $y = a^x$, then find $\frac{dy}{dx}$ using the Chain Rule. (Hint: Use the fact that $a = e^{\ln a}$.)

Example 3.23. Find the derivative of $y = 10^{1-x^2}$. 
3.5 Implicit Differentiation

Previously, we always took the derivative of functions that were in the form of $f(x) = y$, but now we look at equations like $x^2 + y^2 = 16$ and $x^3 + y^3 = 6xy$. How do we handle these? How can we find derivatives for functions not of the form $f(x) = y$? Fortunately we use the method of implicit differentiation.

**Implicit Differentiation**

<table>
<thead>
<tr>
<th>Step 1</th>
<th>Differentiate both sides of the equation with respect to $x$, or the independent variable.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 2</td>
<td>Use quotient, product, and chain rule where necessary.</td>
</tr>
<tr>
<td>Step 3</td>
<td>Solve for $\frac{dy}{dx}$ or $y'$.</td>
</tr>
</tbody>
</table>

**Example 3.24.** Use implicit differentiation to find $\frac{dy}{dx}$.

a) $2\sqrt{x} + \sqrt{y} = 3$

b) $y^5 + x^2 y^3 = 1 + ye^{x^2}$
c) \( y \sin(x^2) = x \sin(y^2) \)

d) \( \tan(x - y) = \frac{y}{1 + x^2} \)
Example 3.25. Use implicit differentiation to find an equation of the tangent line to $x^2 + 2xy - y^2 + x = 2$ at $(1, 2)$. 
3.5.1 Derivatives of Inverse Trigonometric Functions

<table>
<thead>
<tr>
<th>Derivatives of Inverse Trigonometric Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}} )</td>
</tr>
<tr>
<td>( \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}} )</td>
</tr>
</tbody>
</table>

Example 3.26. Prove

\[ \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}} \]

Example 3.27. Prove

\[ \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2 - 1}} \]
Example 3.28. Find the derivative of the function and simplify.

\(a) \quad y = \sqrt{\tan^{-1} x} \quad \quad \quad \quad \quad \quad \quad b) \quad g(x) = \sqrt{x^2 - 1} \sec^{-1} x\)
3.6 Derivatives of Logarithmic Functions

\[
\frac{d}{dx} (\log_a x) = \frac{1}{x \ln a} \quad \frac{d}{dx} (\ln x) = \frac{1}{x}
\]

**Example 3.29.** Prove \( \frac{d}{dx} (\log_a x) = \frac{1}{x \ln a} \).

**Note.** If we replace \( a = e \), notice we obtain directly the derivative for \( \ln x \).

**Example 3.30.** Differentiate.

a) \( f(x) = \ln(x^2 + 10) \) \quad b) \( g(x) = \log_5(xe^x) \)
c) \( y = \frac{1}{\ln x} \)  

d) \( y = \log_2 (e^{-x} \cos \pi x) \)

**Example 3.31.** Find an equation of the tangent line to \( y = \ln (x^3 - 7) \) at \((0, 2)\).
3.6.1 Logarithmic Differentiation

<table>
<thead>
<tr>
<th>Logarithmic Differentiation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Step 1.</strong> Take natural logarithms of both sides of an equation and simplify using the Laws of Logarithms.</td>
</tr>
<tr>
<td><strong>Step 2.</strong> Differentiate implicitly with respect to the independent variable.</td>
</tr>
<tr>
<td><strong>Step 3.</strong> Solve for ( \frac{dy}{dx} ) or ( y' ).</td>
</tr>
</tbody>
</table>

**Example 3.32.** Prove the Power Rule: If \( f(x) = x^n \), then \( f'(x) = nx^{n-1} \), using logarithmic differentiation.
Example 3.33. Use logarithmic differentiation to find the derivative of the function.

a) $y = x^{\cos x}$

b) $y = \sqrt{x} e^{x^2} (x^2 + 1)^{10}$
3.7 Rates of Change

Recall, in Section 2.7, \( s = f(t) \) was the position function and the instantaneous velocity was \( \frac{ds}{dt} \), the rate of change of displacement with respect to time. Hence, given \( s = f(t) \), \( v(t) = s'(t) \). Furthermore, the instantaneous rate of change of velocity with respect to time, \( \frac{dv}{dt} \), is called \textit{acceleration}, \( a(t) \). Thus, \( a(t) = v'(t) = s''(t) \).

**Example 3.34.** A particle moves according to the law of motion \( s = f(t) = te^{-t}, \ t \geq 0 \), where \( t \) is measured in seconds and \( s \) in feet.

\[ a) \text{ Find the velocity at time } t. \quad b) \text{ What is the velocity after 3 seconds?} \]

\[ c) \text{ When is the particle at rest?} \quad d) \text{ When is the particle moving in a positive direction?} \]
3.7. **RATES OF CHANGE**

**e)** Find the total distance traveled in the first 8 seconds.

**f)** Find the acceleration at time $t$ and after 3 seconds.

**g)** Sketch $s$, $v$, and $a$ for $0 \leq t \leq 8$.

**h)** When is the particle speeding up? When is it slowing down?
Example 3.35. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s. Find the rate at which the area within the circle is increasing after 1 second, 3 seconds, and 5 seconds. What can you conclude?
3.8 Exponential Growth and Decay

If $y(t)$ is the value of a quantity $y$ at time $t$ and if the rate of change of $y$ with respect to $t$ is proportional to its size $y(t)$ at any time, then

$$\frac{dy}{dt} = ky,$$

where $k$ is a constant. This equation is called a **differential equation**—sometimes called **law of natural growth** ($k > 0$) or **law of natural decay** ($k < 0$).

**Theorem 3.8.1.** The only solutions of the differential equation $\frac{dy}{dt} = ky$ are the exponential functions

$$y(t) = y(0)e^{kt}$$

**Example 3.36.** A bacteria culture grows with a constant relative growth rate. After 2 hours there are 600 bacteria and after 8 hours the count is 75,000.

a) Find the initial population.
b) Find an expression for the population after $t$ hours.

c) Find the number of cells after 5 hours.

d) Find the rate of growth after 5 hours.

e) When will the population reach 200,000?
Example 3.37. A sample of tritium-3 decayed to 94.5% of its original amount after a year.

a) What is the half-life of tritium-3?  
b) How long would it take the sample to decay to 20% of its original amount?
3.8.1 Newton’s Law of Cooling

If $T(t)$ is the temperature of an object at time $t$ and $T_s$ is the temperature of the surroundings, then

$$\frac{dT}{dt} = k(T - T_s),$$

where $k$ is a constant.

**Note.** Let $y(t) = T(t) - T_s$ so that $y'(t) = T'(t)$ and

$$\frac{dy}{dt} = ky$$

Hence, we can use the previous method to find an expression for $T$.

**Example 3.38.** A thermometer is taken from a room where the temperature is 20°C to the outdoors, where the temperature is 5°C. After one minute the thermometer reads 12°C.

**a)** What will the reading be after one more minute?

**b)** When will the thermometer read 6°C?
3.9 Related Rates

In related rates problems, the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity. Then we use the Chain Rule to differentiate both sides with respect to time. *The key thing to remember is that rate of changes are derivatives.*

**Example 3.39.** A spherical balloon is expanding. Given that the radius is increasing at a rate of 2 inches per minute, at what rate is the volume increasing when the radius is 5 inches?
Example 3.40. A 13-foot ladder leans against the side of a building. The bottom of the ladder slides away from the building at a rate of 0.1 feet per second.

a) How fast is the top of the ladder sliding down the building when the top of the ladder is 5 feet above the ground?

b) How fast is the angle between the ladder and the ground changing when the top of the ladder is 5 feet above the ground?
Example 3.41. A cylindrical tank with radius 5 m is being filled with water at a rate of 3 m$^3$/min. How fast is the height of the water increasing?

Example 3.42. A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?
3.10 Linear Approximations & Differentials

*Recall.* In Section 2.7, we’ve seen that a curve lies very close to its tangent line near the point of tangency. We use this observation for a method of finding approximate values of functions.

### 3.10.1 Linear Approximation

Suppose that \( f \) is differentiable at \( x = a \). Then the equation of the tangent line at the point \((a, f(a))\) is

\[
y = f'(a)(x - a) + f(a)
\]

This gives an approximation of a function using tangent lines for values of \( x \) close to \( a \), i.e.,

\[
f(x) \approx f'(a)(x - a) + f(a).
\]

This is called the *linear approximation* or *tangent line approximation* of \( f \) at \( a \).

**Note.** The equation of a tangent line gives exact values at \( x = a \), but the linear approximation gives approximate values for the original function \( f(x) \).

**Example 3.43.** Find the linear approximation of the function \( g(x) = \sqrt{1+x} \) at \( a = 0 \) and use it to approximate the numbers \( \sqrt{0.95} \) and \( \sqrt{1.1} \).
3.10.2 Differentials

Let \( y = f(x) \) represent a function that is differentiable on an open interval containing \( x \). The differential of \( x \), denoted by \( dx \), is any nonzero real number. The differential of \( y \), denoted by \( dy \) is given by

\[
dy = f'(x) \, dx
\]

**Note.** The geometric meaning of differentials is presented as

\[
\Delta y = f(x + \Delta x) - f(x)
\]

We can also say that \( \Delta y \approx dy \) or \( \Delta y \approx f'(x) \, dx \) which may be used to approximate the change in \( y \).

**Example 3.44.** Let \( y = \cos x \). Find the differential \( dy \) and evaluate \( dy \) for \( x = \pi/3 \) and \( dx = 0.05 \).

**Example 3.45.** Let \( y = x^2 + 2x \). Find the value of \( dy \) when \( x = 1 \) and \( dx = \Delta x = 0.01 \). Compare the result with \( \Delta y \) for \( x = 1 \).
Example 3.46. Use a linear approximation (or differentials) to estimate $e^{-0.015}$.

Example 3.47. The radius of a ball bearing is measured to be 0.5-inch. If the correct measurement is within 0.01-inch, estimate the maximum error, relative error, and the percentage error in the volume of the ball bearing.
3.11 Hyperbolic Functions

Combinations of the functions $e^x$ and $e^{-x}$ arise so frequently in mathematics that they have their own name called \textit{hyperbolic functions}. We will see in this section why they are called hyperbolic functions.

\begin{tabular}{|c|c|}
\hline
\textbf{Hyperbolic Functions} & \\
\hline
$\sinh x = \frac{e^x - e^{-x}}{2}$ & $\cosh x = \frac{1}{\sinh x}$ \\
$\cosh x = \frac{e^x + e^{-x}}{2}$ & $\sinh x = \frac{1}{\cosh x}$ \\
$tanh x = \frac{\sinh x}{\cosh x}$ & $\coth x = \frac{\cosh x}{\sinh x}$ \\
\hline
\end{tabular}

\begin{tabular}{|c|c|}
\hline
\textbf{Hyperbolic Identities} & \\
\hline
$\sinh (-x) = -\sinh x$ & $\cosh (-x) = \cosh x$ \\
$\cosh^2 x - \sinh^2 x = 1$ & $1 - \tanh^2 x = \text{sech}^2 x$ \\
$\sinh (x + y) = \sinh x \cosh y + \cosh x \sinh y$ & $\cosh (x + y) = \cosh x \cosh y + \sinh x \sinh y$ \\
\hline
\end{tabular}
Example 3.48. Prove the identity $\cosh^2 x - \sinh^2 x = 1$.

Example 3.49. Prove the identity $\sinh 2x = 2 \sinh x \cosh x$.

Example 3.50. If we let $x = \cosh t$ and $y = \sinh t$, use the identity $\cosh^2 x - \sinh^2 x = 1$ to show that the graph is a hyperbola. (Think of this as you did with the unit circle and trigonometric functions.) Sketch a diagram.
3.11. Derivatives of Hyperbolic Functions

Example 3.51. Find the derivative for \( y = \cosh x \).

\[
\begin{align*}
\frac{d}{dx}(\sinh x) &= \cosh x \\
\frac{d}{dx}(\csch x) &= -\csch x \coth x \\
\frac{d}{dx}(\cosh x) &= \sinh x \\
\frac{d}{dx}(\sech x) &= -\sech x \tanh x \\
\frac{d}{dx}(\tanh x) &= \sech^2 x \\
\frac{d}{dx}(\coth x) &= -\csch^2 x 
\end{align*}
\]

Example 3.52. Find the derivative for \( f(t) = \sech^2(e^t) \).
3.11.2 Inverse Hyperbolic Functions

We can see from before that sinh and tanh are one-to-one, so their inverses exist. Since cosh is not one-to-one, we restrict the domain for cosh to $[0, \infty)$.

\[
\begin{align*}
    y &= \sinh^{-1} x \iff \sinh y = x \\
    y &= \cosh^{-1} x \iff \cosh y = x \quad \text{and} \quad y \geq 0 \\
    y &= \tanh^{-1} x \iff \tanh y = x
\end{align*}
\]

Since hyperbolic functions are defined with exponential functions, it makes sense that their inverses are represented with natural logarithmic functions.

\[
\begin{align*}
    \sinh^{-1} x &= \ln \left( x + \sqrt{x^2 + 1} \right), \text{ where } x \in \mathbb{R} \\
    \cosh^{-1} x &= \ln \left( x + \sqrt{x^2 - 1} \right), \text{ where } x \geq 1 \\
    \tanh^{-1} x &= \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right), \text{ where } -1 < x < 1
\end{align*}
\]
Example 3.53. Show that $\cosh^{-1} x = \ln (x + \sqrt{x^2 - 1})$. 
3.11.3 Derivatives of Inverse Hyperbolic Functions

\[
\begin{align*}
\frac{d}{dx} (\sinh^{-1} x) &= \frac{1}{\sqrt{1 + x^2}} & \frac{d}{dx} (\cosh^{-1} x) &= \frac{1}{x \sqrt{1 + x^2}} \\
\frac{d}{dx} (\cosh^{-1} x) &= \frac{1}{\sqrt{x^2 - 1}} & \frac{d}{dx} (\sech^{-1} x) &= -\frac{1}{x \sqrt{1 - x^2}} \\
\frac{d}{dx} (\tanh^{-1} x) &= \frac{1}{1 - x^2} & \frac{d}{dx} (\coth^{-1} x) &= \frac{1}{1 - x^2}
\end{align*}
\]

Example 3.54. Prove that \( \frac{d}{dx} (\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}} \).
Example 3.55. Find the derivative and simplify $y = x^2 \sinh^{-1}(2x)$. 
Chapter 4

Applications of Differentiation

4.1 Maximum & Minimum Values

<table>
<thead>
<tr>
<th>Definition.</th>
</tr>
</thead>
<tbody>
<tr>
<td>• A function $f$ has an <strong>absolute maximum</strong> (global maximum) at $c$ if $f(c) \geq f(x)$ for $x$ in $D$, the domain of $f$. The number of $f(c)$ is called the maximum value of $f$ on $D$.</td>
</tr>
<tr>
<td>• A function $f$ has an <strong>absolute minimum</strong> (global minimum) at $c$ if $f(c) \leq f(x)$ for $x$ in $D$, the domain of $f$. The number of $f(c)$ is called the minimum value of $f$ on $D$.</td>
</tr>
<tr>
<td>• A function $f$ has a <strong>local maximum</strong> (or relative maximum) at $c$ if $f(c) \geq f(x)$ when $x$ is near $c$.</td>
</tr>
<tr>
<td>• A function $f$ has a <strong>local minimum</strong> (or relative minimum) at $c$ if $f(c) \leq f(x)$ when $x$ is near $c$.</td>
</tr>
</tbody>
</table>
Example 4.1. Graph $f(x) = 1 + (x + 1)^2$ on the interval $-2 \leq x < 5$ and use the graph to determine the absolute and local maximum and minimum values of $f$.

The Extreme Value Theorem If $f$ is continuous on a closed interval $[a, b]$, then $f$ attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c$ and $d$ in $[a, b]$. 
4.1. MAXIMUM & MINIMUM VALUES

Fermat’s Theorem If $f$ has a local maximum or minimum at $c$, and if $f’(c)$ exists, then $f’(c) = 0$.

Example 4.2. If $f(x) = x^3$, then find $f’(x) = 0$. What does this say about Fermat’s Theorem?

Definition. A critical number of a function $f$ is a number $c$ in the domain of $f$ such that either $f’(c) = 0$ or $f’(c)$ does not exist.

Note. If $f$ has a local maximum or minimum at $c$, then $c$ is a critical number of $f$. 
Example 4.3. Find the critical numbers of the function \( g(x) = x^{1/3} - x^{-2/3} \).

The Closed Interval Method To find absolute maximum and minimum values of a continuous function \( f \) on a closed interval \([a, b]\):

Step 1. Find the values of \( f \) at the critical numbers of \( f \) in \([a, b]\).

Step 2. Find the values of \( f \) at the endpoints of \( f \).

Step 3. Using all values of \( f \) from the previous steps, we say the largest value of \( f \) is the absolute maximum and the smallest value of \( f \) is the absolute minimum.

Example 4.4. Find the absolute maximum and absolute minimum values of \( f \) on the given interval.

a) \( f(x) = x^3 - 3x + 1, [0, 3] \)
b) \( f(x) = \frac{x^2 - 4}{x^2 + 4}, [-4, 4] \)

c) \( f(t) = t + \cot\left(\frac{t}{2}\right), \left[\frac{\pi}{4}, \frac{7\pi}{4}\right] \)
4.2 The Mean Value Theorem

**Rolle’s Theorem** Let \( f \) be a function that satisfies the following three hypotheses:

1. \( f \) is continuous on the closed interval \([a, b]\).
2. \( f \) is differentiable on the open interval \((a, b)\).
3. \( f(a) = f(b) \)

Then there is a number \( c \) in \((a, b)\) such that \( f'(c) = 0 \).

**Example 4.5.** Verify that \( f(x) = x^3 - x^2 - 6x + 2 \) satisfies the three hypotheses of Rolle’s Theorem on the interval \([0, 3]\). Then find all numbers \( c \) that satisfy the conclusion of Rolle’s Theorem.
The Mean Value Theorem

Let \( f \) be a function that satisfies the following hypotheses:

1. \( f \) is continuous on the closed interval \([a, b]\).
2. \( f \) is differentiable on the open interval \((a, b)\).

Then there is a number \( c \) in \((a, b)\) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

or equivalently

\[
f(b) - f(a) = f'(c)(b - a)
\]

Example 4.6. Verify that \( f(x) = x^3 + x - 1 \) satisfies the hypotheses of Mean Value Theorem on the interval \([0, 2]\). Then find all numbers \( c \) that satisfy the conclusion of the Mean Value Theorem.
Example 4.7. Show that the equation $2x - 1 - \sin x = 0$ has exactly one real root.
Example 4.8. Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of $x$. How large can $f(2)$ possibly be?
4.3 How Derivatives Affect the Shape of a Graph

4.3.1 What does $f'$ say about $f$?

- If $f'(x) > 0$ on an interval, then $f$ is increasing on that interval.
- If $f'(x) < 0$ on an interval, then $f$ is decreasing on that interval.

**The First Derivative Test** Suppose that $c$ is a critical number of a continuous function $f$.

- If $f'$ changes from positive to negative at $c$, then $f$ has a local maximum at $c$.
- If $f'$ changes from negative to positive at $c$, then $f$ has a local minimum at $c$.
- If $f'$ does not change signs at $c$, then $f$ has a no local maximum or minimum at $c$.

**Example 4.9.** Determine where $f(x) = 4x^3 + 3x^2 - 6x + 1$ is increasing and/or decreasing, and local extrema (maxima or minima).
4.3.2 What does $f''$ say about $f$?

- If the graph of $f$ lies above all its tangents on an interval $I$, then it is called **concave upward** on $I$.
- If the graph of $f$ lies below all its tangents on an interval $I$, then it is called **concave downward** on $I$.

**Concavity Test**

- If $f''(x) > 0$ for all $x$ in $I$, then the graph of $f$ is concave upward on $I$.
- If $f''(x) < 0$ for all $x$ in $I$, then the graph of $f$ is concave downward on $I$.

**Definition.** A point $P$ on a curve $y = f(x)$ is called an **inflection point** if $f$ is continuous there and the curve changes concavity at $P$. 
Example 4.10. Sketch a graph of a function that satisfies all the given conditions.

\[ f'(x) > 0 \text{ for all } x \neq 1, \text{ vertical asymptote } x = 1 \]
\[ f''(x) > 0 \text{ if } x < 1 \text{ or } x > 3, \quad f''(x) < 0 \text{ if } 1 < x < 3 \]

**Second Derivative Test** Suppose \( f''(x) \) is continuous near \( c \).

- If \( f'(c) = 0 \) and \( f''(c) > 0 \), then \( f \) has a local minimum at \( c \).
- If \( f'(c) = 0 \) and \( f''(c) < 0 \), then \( f \) has a local maximum at \( c \).
Example 4.11. Find all local extrema using the Second Derivative Test for $f(x) = x^5 - 5x + 3$. Find all intervals of increasing/decreasing, points of inflection, and intervals of concavity. Verify your results with a graphing calculator.
Example 4.12. Find all local extrema using the Second Derivative Test for $g(t) = t + \cos t$, $-2\pi \leq t \leq 2\pi$. Find all local extrema and intervals of increasing/decreasing. Verify your results with a graphing calculator.
4.4 Indeterminate Form and L’Hospital’s Rule

Recall. Back in Chapter 2, we were to find limits of functions like

\[
\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} \quad \text{and} \quad \lim_{x \to 0} \frac{\sin x}{x} \quad \text{and} \quad \lim_{x \to \infty} \frac{x^2 - 1}{2x^2 + 1}
\]

where we reduced or looked at the function geometrically.

**Example 4.13.** What if we were given \( \lim_{x \to 1} \frac{\ln x}{x - 1} \)? How can we evaluate this limit?

---

**L’Hospital’s Rule** Suppose \( f \) and \( g \) are differentiable and \( g'(x) \neq 0 \) on an open interval \( I \) that contains \( a \) (except possible at \( a \)). Suppose that

\[
\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0
\]

or that

\[
\lim_{x \to a} f(x) = \pm \infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm \infty,
\]

i.e., we obtain an indeterminate form of type \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \). Then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
\]

if the limit on the right side exists.
Example 4.14. Find \( \lim_{x \to 1} \frac{\ln x}{x - 1} \).

Example 4.15. Find \( \lim_{x \to 1} \frac{x^9 - 1}{x^5 - 1} \).

Example 4.16. Find \( \lim_{\theta \to \pi/2} \frac{1 - \sin \theta}{\csc \theta} \).

Example 4.17. Find \( \lim_{x \to \infty} \frac{e^x}{x^2} \).
4.4.1 Indeterminate Products

If \( \lim_{{x \to a}} f(x) = 0 \) and \( \lim_{{x \to a}} g(x) = \infty \), then

\[
\lim_{{x \to a}} f(x) \cdot g(x) = 0 \cdot \infty
\]

is called an indeterminate form of type \( 0 \cdot \infty \).

We re-write the product as a quotient:

\[
f g = \frac{f}{1/g} \quad \text{or} \quad \frac{g}{1/f}
\]

**Example 4.18.** Evaluate \( \lim_{{x \to 0^+}} x \ln x \).

**Example 4.19.** Evaluate \( \lim_{{x \to -\infty}} x^2 e^x \).
4.4.2 Indeterminate Differences

If \( \lim_{x \to a} f(x) = \infty \) and \( \lim_{x \to a} g(x) = \infty \), then

\[
\lim_{x \to a} [f(x) - g(x)] = \infty - \infty
\]

is called an indeterminate form of type \( \infty - \infty \).

We re-write the difference as a quotient.

**Example 4.20.** Find \( \lim_{x \to 0} (\csc x - \cot x) \).

**Example 4.21.** Find \( \lim_{x \to \infty} (xe^{1/x} - x) \).
4.4.3 Indeterminate Powers

Several indeterminate forms arise from the limit

\[ \lim_{x \to a} [f(x)]^{g(x)} \]

1. \( \lim_{x \to a} f(x) = 0 \) and \( \lim_{x \to a} g(x) = 0 \) type \( 0^0 \)

2. \( \lim_{x \to a} f(x) = \infty \) and \( \lim_{x \to a} g(x) = 0 \) type \( \infty^0 \)

3. \( \lim_{x \to a} f(x) = 1 \) and \( \lim_{x \to a} g(x) = \pm \infty \) type \( 1^\infty \)

We re-write \( f(x)^{g(x)} \) using natural logarithms. Let \( y = [f(x)]^{g(x)} \) and then \( \ln y = g(x) \cdot \ln f(x) \).

**Example 4.22.** Find \( \lim_{x \to 0} (1 - 2x)^{1/x} \).
4.5 Summary of Curve Sketching

Example 4.23. Sketch the graph of \( y = x^3 + 6x^2 + 9x \) by following all steps below without using a calculator.

a) Find the domain. 

b) Determine any asymptotes.

c) Determine any symmetry. 

d) Find intercepts.
4.5. SUMMARY OF CURVE SKETCHING

e) Determine intervals of increasing or decreasing.

f) Determine any local extrema.

g) Determine concavity and points of inflection.

h) Sketch the curve.
Example 4.24. Sketch the graph of $y = \frac{x}{x - 1}$ by following all steps below without using a calculator.

a) Find the domain.  

b) Determine any asymptotes.

c) Determine any symmetry.  

d) Find intercepts.
4.5. SUMMARY OF CURVE SKETCHING

**e)** Determine intervals of increasing or decreasing.

**f)** Determine any local extrema.

**g)** Determine concavity and points of inflection.

**h)** Sketch the curve.
Example 4.25. Sketch the graph of \( y = \sec x + \tan x \), \( 0 < x < \pi/2 \), by following all steps below without using a calculator.

a) Find the domain.

b) Determine any asymptotes.

c) Determine any symmetry.

d) Find intercepts.
e) Determine intervals of increasing or decreasing.

f) Determine any local extrema.

g) Determine concavity and points of inflection.

h) Sketch the curve.
4.6 Graphing with Calculus & Calculators

Example 4.26. Sketch a graph of $f$. Using the graphs of $f'$ and $f''$, estimate the intervals of increasing/decreasing, local extrema, intervals of concavity, and inflections points.

$$f(x) = 4x^4 - 32x^3 + 89x^2 - 95x + 29$$
Example 4.27. Sketch a graph of $f$. Using the graphs of $f'$ and $f''$, estimate the intervals of increasing/decreasing, local extrema, intervals of concavity, and inflection points.

$$f(x) = x^2 - 4x + 7 \cos x, \quad -4 \leq x \leq 4$$
4.7 Optimization Problems

When we are solving optimization problems, i.e., maximizing or minimizing functions, we

**Step 1.** Read and understand the problem.

**Step 2.** Draw a picture.

**Step 3.** Write a function. *The goal will be to write the equation in terms of one variable.*

**Step 4.** Take the derivative and find any critical numbers.

**Step 5.** Verify the critical number(s) is a maximum or minimum by using the First or Second Derivative Test.

**Step 6.** Answer the question.

**Example 4.28.** Find the dimensions of a rectangle with area 1000 square meters whose perimeter is as small as possible.
Example 4.29. A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each corner and bending up the sides. Find the largest volume that such a box can have.
Example 4.30. A rectangular storage container with an open top is to have a volume of 10 cubic meters. The length of its base is twice the width. Material for the base costs $10 per square meter and the material for the sides costs $6 per square meter. Find the cost of materials for the cheapest such container.
Example 4.31. Find the point on the line $6x + y = 9$ that is closest to the point $(-3, 1)$. 
Example 4.32. A piece of wire 10 meters long is cut into two pieces. One piece is bent into a square and the other a circle. How should the wire be cut so that the total area is a maximum? A minimum?
4.9 Antiderivatives

What if a physicist knows the velocity, but wants to know the position? Or a biologist who knows the rate at which the population is increasing, but wants to know the population? In any case, what if we are given the first derivative function $f$, but desire to have the original function $F$, too? When this happens, then we say $F$ is the antiderivative of $f$.

**Definition.** A function $F$ is called an antiderivative of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x$ in $I$.

**Example 4.33.** Let’s look at $f(x) = x^2$. What are the possibilities for $F(x)$?

**Theorem 4.9.1.** If $F$ is an antiderivative of $f$ on an interval $I$, then the most general antiderivative of $f$ on $I$ is

$$F(x) + C,$$

where $C$ is an arbitrary constant.

**Example 4.34.** Find the most general antiderivative of each function.

a) $f(x) = \sec^2 x$

b) $f(x) = \frac{1}{x}$

c) $f(x) = x^n$, where $n \neq -1$
<table>
<thead>
<tr>
<th>Function</th>
<th>Particular Antiderivative</th>
<th>Function</th>
<th>Particular Antiderivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( cf(x) )</td>
<td>( cF(x) )</td>
<td>( \sin x )</td>
<td>( -\cos x )</td>
</tr>
<tr>
<td>( f(x) + g(x) )</td>
<td>( F(x) + G(x) )</td>
<td>( \cos x )</td>
<td>( \sin x )</td>
</tr>
<tr>
<td>( x^n, n \neq -1)</td>
<td>( \frac{x^{n+1}}{n+1} )</td>
<td>( \sec^2 x )</td>
<td>( \tan x )</td>
</tr>
<tr>
<td>( \frac{1}{x} )</td>
<td>( \ln</td>
<td>x</td>
<td>)</td>
</tr>
<tr>
<td>( e^x )</td>
<td>( e^x )</td>
<td>( \frac{1}{\sqrt{1-x^2}} )</td>
<td>( \sin^{-1} x )</td>
</tr>
<tr>
<td>( C, \text{ constant} )</td>
<td>( Cx )</td>
<td>( \frac{1}{1+x^2} )</td>
<td>( \tan^{-1} x )</td>
</tr>
</tbody>
</table>

**Example 4.35.** Find the most general antiderivative of \( g(x) = \frac{5 - 4x^3 + 2x^6}{x^6} \).

**Example 4.36.** Find \( f \) if \( f''(x) = 6x + \sin x \).
Example 4.37. Find $f$ if $f'(x) = \frac{4}{\sqrt{1 - x^2}}$ and $f\left(\frac{1}{2}\right) = 1$.

Example 4.38. A particle is moving with the given acceleration $a(t) = \cos t + \sin t$, where $s(0) = 0$ and $v(0) = 5$. Find the position function.
Chapter 5

Integrals

5.1 Areas and Distances

Areas of defined shapes, like rectangles or triangles, are easy to find. Area under a curve is a little more involved. We must cut the region into rectangles and then take the limit of the areas of these rectangles as we increase the number of rectangles.

Example 5.1. Use rectangles to estimate the area under the parabola \( y = x^2 \) from 0 to 1.

![Graph of the parabola y = x^2 from 0 to 1]
**Example 5.2.** Now use rectangles with left end points to estimate the area under the parabola \( y = x^2 \) from 0 to 1. Compare the results when the right endpoints were used.
Example 5.3. Find the area of a general region $S$ by subdividing $S$ into $n$ strips on an interval $[a, b]$.

Definition. The \textit{area} $A$ of the region $S$ that lies under the graph of the continuous function $f$ is the limit of the sum of the areas of approximating rectangles:

\[
A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \left[ f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \right] = \lim_{n \to \infty} \frac{n}{\Delta x} \sum_{i=1}^{n} f(x_i)\Delta x
\]

or similarly

\[
A = \lim_{n \to \infty} L_n = \lim_{n \to \infty} \left[ f(x_0)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x \right] = \lim_{n \to \infty} \frac{n}{\Delta x} \sum_{i=1}^{n} f(x_{i-1})\Delta x
\]
Example 5.4. Estimate the area under the graph of \( f(x) = \sqrt{x} \) from \( x = 0 \) to \( x = 4 \) using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or overestimate? Next repeat the example using left endpoints.
Example 5.5. Oil leaked from a tank at a rate of $r(t)$ liters per hour. The rate decreased as time passed and values of the rate at two-hour time intervals are shown in the table below. Find lower and upper estimates for the total amount of oil that leaked out.

<table>
<thead>
<tr>
<th>$t$ hours</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r(t)$ L/h</td>
<td>8.7</td>
<td>7.6</td>
<td>6.8</td>
<td>6.2</td>
<td>5.7</td>
<td>5.3</td>
</tr>
</tbody>
</table>

5.2 The Definite Integrals

**Definite Integral:** If \( f \) is a function defined for \( a \leq x \leq b \), we divide the interval \([a, b]\) into \( n \) subintervals of equal width \( \Delta x = \frac{b - a}{n} \). We let \( x_0 (= a), x_1, x_2, \ldots, x_n (= b) \) be the endpoints of these subintervals and we let \( x_1^*, x_2^*, \ldots, x_n^* \) be any sample points in these subintervals, so \( x_i^* \) lies in the \( i \)th subinterval. Then the **definite integral** of \( f \) from \( a \) to \( b \) is

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x
\]

provided that this limit exists. If it does exist, we say that \( f \) is **integrable** on \([a, b]\).

**Note.** The symbol \( \int \) is called the **integral sign**, \( f(x) \) is called the **integrand**, \( a \) and \( b \) are called the **limits of integration**, and \( dx \) indicates that the independent variable is \( x \); \( a \) is called the lower limit and \( b \) is called the upper limit. The sum \( \sum_{i=1}^{n} f(x_i^*) \Delta x \) is called the **Riemann sum**.

**Remark.** The definite integral \( \int_{a}^{b} f(x) \, dx \) is a number; it does not depend on \( x \). We could easily change \( x \) without changing the value of the integral, e.g.,

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(t) \, dt = \int_{a}^{b} f(r) \, dr
\]

**Example 5.6.** If \( f(x) = x^2 - 2x \), \( 0 \leq x \leq 3 \), evaluate the Riemann sum with \( n = 6 \), taking the sample points to be right endpoints. What does the Riemann sum represent? Draw a diagram.
In general, if \( f \) is integrable on \([a, b]\), then

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x
\]

where \( \Delta x = \frac{b - a}{n} \) and \( x_i = a + i \Delta x \).

**Example 5.7.** Express \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{\cos x_i}{x_i} \Delta x \) as a definite integral on \([\pi, 2\pi]\).

<table>
<thead>
<tr>
<th>Summation Formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} )</td>
</tr>
<tr>
<td>( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} )</td>
</tr>
<tr>
<td>( \sum_{i=1}^{n} i^3 = \left[ \frac{n(n+1)}{2} \right]^2 )</td>
</tr>
</tbody>
</table>
Example 5.8. Use the form of the definition of the integral to evaluate

$$\int_{1}^{4} (x^2 + 2x - 5) \, dx$$
Example 5.9. Prove $\int_a^b x^2 \,dx = \frac{b^3 - a^3}{3}$. 
Example 5.10. The function $g(x)$ is graphed below. Evaluate each integral by interpreting it in terms of areas: $\int_0^2 g(x) \, dx$, $\int_6^8 g(x) \, dx$, and $\int_0^9 g(x) \, dx$. 

![Graph of function g(x)](image)
5.3 The Fundamental Theorem of Calculus

5.3.1 Properties of the Definite Integral

<table>
<thead>
<tr>
<th>Properties of the Integral:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \int_{a}^{b} f(x) , dx = - \int_{b}^{a} f(x) , dx )</td>
</tr>
<tr>
<td>2. ( \int_{a}^{a} f(x) , dx = 0 )</td>
</tr>
<tr>
<td>3. ( \int_{a}^{b} c , dx = c(b - a) )</td>
</tr>
<tr>
<td>4. ( \int_{a}^{b} e^x , dx = e^b - e^a )</td>
</tr>
<tr>
<td>5. ( \int_{a}^{b} [f(x) \pm g(x)] , dx = \int_{a}^{b} f(x) , dx \pm \int_{a}^{b} g(x) , dx )</td>
</tr>
<tr>
<td>6. ( \int_{a}^{b} cf(x) , dx = c \int_{a}^{b} f(x) , dx )</td>
</tr>
<tr>
<td>7. ( \int_{a}^{c} f(x) , dx + \int_{c}^{b} f(x) , dx = \int_{a}^{b} f(x) , dx )</td>
</tr>
</tbody>
</table>

Example 5.11. Prove the property \( \int_{a}^{b} [f(x) + g(x)] \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx \). (Hint: Use the definition of the integral.)
Example 5.12. Use the properties of integrals to evaluate $\int_{1}^{3} (2e^x - 1) \, dx$.

Example 5.13. If $\int_{1}^{5} f(x) \, dx = 12$ and $\int_{4}^{5} f(x) \, dx = 3.6$, find $\int_{1}^{4} f(x) \, dx$. 
Example 5.14. Given the graph and $g(x) = \int_0^x f(t) \, dt$, find values of $g(0), g(1), g(3), g(4)$, and $g(5)$.

The Fundamental Theorem of Calculus, Part I: If $f$ is continuous on $[a, b]$, then the function $g$ defined by

$$g(x) = \int_a^x f(t) \, dt \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on $(a, b)$, and $g'(x) = f(x)$.

Example 5.15. Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the function.

a) $g(x) = \int_3^x e^{t^2-x} \, dt$  

b) $g(r) = \int_0^r \sqrt{x^2+4} \, dx$  

c) $G(x) = \int_x^1 \cos \sqrt{t} \, dt$
The Fundamental Theorem of Calculus, Part 2: If \( f \) is continuous on \([a, b]\), then
\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]
where \( F \) is any antiderivative of \( f \), that is, a function such that \( F' = f \).

Example 5.16. Evaluate the integral.

a) \( \int_{-2}^{5} 6 \, dx \)  
b) \( \int_{0}^{1} (3 + x\sqrt{x}) \, dx \)

c) \( \int_{0}^{\pi/4} \sec \theta \tan \theta \, d\theta \)  
d) \( \int_{0}^{1} \frac{4}{t^2 + 1} \, dt \)
Example 5.17. Evaluate $\int_0^\pi f(x) \, dx$ where $f(x) = \begin{cases} \sin x & \text{if } 0 \leq x < \pi/2 \\ \cos x & \text{if } \pi/2 \leq x \leq \pi \end{cases}$. 
5.4 Indefinite Integrals & The Net Change Theorem

The Indefinite Integral: The *indefinite integral* is given as

$$\int f(x) \, dx = F(x) \quad \text{means} \quad F'(x) = f(x)$$

where the indefinite integral is used as the most general antiderivative of $f$.

**Note.** Do not confuse the two types on integrals! The *indefinite integral*, $\int f(x) \, dx$, is a family of functions. The *definite integral* is a value, i.e., a number.

**Example 5.18.** Find the general indefinite integral.

a) $\int (\sqrt{x^3} + \sqrt[3]{x^2}) \, dx$

b) $\int \left(x^2 + 1 + \frac{1}{x^2 + 1}\right) \, dx$
5.4. INDEFINITE INTEGRALS & THE NET CHANGE THEOREM

\[ c) \int (\csc^2 t - 2t^4) \, dt \quad \text{d) } \int \frac{\sin 2x}{\sin x} \, dx \]

**Example 5.19.** Evaluate the integral.

\[ a) \int_1^3 (1 + 2x - 4x^3) \, dx \quad \text{b) } \int_0^5 (2e^x + 4 \cos x) \, dx \]
The Net Change Theorem: The integral of a rate of change is the net change:
\[
\int_a^b F'(x) \, dx = F(b) - F(a)
\]

**Example 5.20.** Given the graph of \( f(t) \) and \( g(x) = \int_a^x f(t) \, dt \), find value of \( g(1) \) if \( g(-4) = 2, A_1 = \int_{-4}^{-2} f(t) \, dt = -2.5, A_2 = \int_{-2}^1 f(t) \, dt = 3, \) and \( A_3 = \int_1^2 f(t) \, dt = -1.25. \)
5.5 The Substitution Rule

We can use the Fundamental Theorem of Calculus to evaluate integrals and the antidifferentiation formulas are useful, but unfortunately, not helpful when trying to integrate functions such as

\[ \int 2x\sqrt{1+x^2} \, dx, \int (x^3 \cos x^4 + 2) \, dx, \text{ or } \int \frac{\ln x}{x} \, dx \]

We discuss, in this section, a method to integrating such functions call the Substitution Rule. This technique is super helpful and should be put in your bag of tricks since you will be integrating a lot more in Chapter 7.

**The Substitution Rule:** If \( u = g(x) \) is a differentiable function whose range is an interval \( I \) and \( f \) is continuous on \( I \), then

\[ \int f(g(x))g'(x) \, dx = \int f(u) \, du \]

**Example 5.21.** Evaluate the integral \( \int x^3(2 + x^4)^5 \, dx \) by making the given substitution \( u = 2 + x^4 \).
Example 5.22. Evaluate the indefinite integral.

a) \[ \int (3t + 2)^4 \, dt \]

b) \[ \int \frac{x}{(x^2 + 1)^2} \, dx \]
c) \[ \int (1 + \tan \theta)^5 \sec^2 \theta \, d\theta \]

d) \[ \int \frac{x^2}{\sqrt{1 - x}} \, dx \]
5.5.1 Definite Integrals

The Substitution Rule for Definite Integrals: If \( g' \) is continuous \([a, b]\) and \( f \) is continuous on the range of \( u = g(x) \), then

\[
\int_{a}^{b} f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du
\]

Example 5.23. Evaluate the definite integral.

a) \( \int_{0}^{\pi/2} \cos x \sin(\sin x) \, dx \)  

b) \( \int_{0}^{1} xe^{-x^2} \, dx \)
5.5.2 Symmetry

**Symmetry:** Suppose $f$ is continuous on $[-a, a]$.

- If $f$ is even [$f(-x) = f(x)$], then $\int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx$.
- If $f$ is odd [$f(-x) = -f(x)$], then $\int_{-a}^{a} f(x) \, dx = 0$.

**Example 5.24.** Evaluate the definite integral.

a) $\int_{-2}^{2} (x^6 + 1) \, dx$

b) $\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4} \, dx$