Chapter 3

Position, Velocity, and Acceleration Revisited

The **position vector** of a particle is a vector drawn from the origin to the location of the particle. In two dimensions:

\[ \vec{r} = x\hat{i} + y\hat{j} \]  

The **displacement vector** is the change in a particle’s position from some time \( t_1 \) to some time \( t_2 \):

\[ \Delta\vec{r} = \vec{r}_f - \vec{r}_i \]

As before, the average velocity of the particle is:

\[ \vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} \]

To find the instantaneous velocity, we take the limit as the time interval goes to zero (as we did in the one dimensional case):
\[ \ddot{\mathbf{v}} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt} \]  

(2)

In two dimensions:

\[ \Delta \mathbf{r} = \mathbf{r}_f - \mathbf{r}_i = (x_f - x_i)\mathbf{i} + (y_f - y_i)\mathbf{j} = \Delta x\mathbf{i} + \Delta y\mathbf{j} \]

\[ \ddot{\mathbf{v}} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta x\mathbf{i} + \Delta y\mathbf{j}}{\Delta t} = \mathbf{\hat{i}} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \mathbf{\hat{j}} \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \mathbf{\hat{i}} \frac{dx}{dt} + \mathbf{\hat{j}} \frac{dy}{dt} \]

The average acceleration is still defined as:

\[ \ddot{\mathbf{a}}_{avg} = \frac{\Delta \ddot{\mathbf{v}}}{\Delta t} \]

To find the instantaneous acceleration, we follow the same process that we used to find the instantaneous velocity:

\[ \ddot{\mathbf{a}} = \lim_{\Delta t \to 0} \frac{\Delta \ddot{\mathbf{v}}}{\Delta t} = \frac{d\ddot{\mathbf{v}}}{dt} \]

If we express the velocity vector in rectangular coordinates:

\[ \ddot{\mathbf{v}} = \mathbf{v}_x \mathbf{\hat{i}} + \mathbf{v}_y \mathbf{\hat{j}} \]

Now the instantaneous acceleration is:
\[ \vec{a} = \frac{d}{dt} \left( v_x \hat{i} + v_y \hat{j} \right) \]
\[ = \frac{dv_x}{dt} \hat{i} + \frac{dv_y}{dt} \hat{j} \]
\[ = \frac{d^2x}{dt^2} \hat{i} + \frac{d^2y}{dt^2} \hat{j} \]

**Example 1:** A particle has a position vector given by
\[ \vec{r} = (30t) \hat{i} + (40t - 5t^2) \hat{j}, \]
where \( r \) is in meters and \( t \) is in seconds. Find the instantaneous velocity and instantaneous acceleration vectors as functions of time. What are the position, velocity, and acceleration of the particle at \( t = 2.00 \) s?

**Solution:**
**Projectile Motion**

Imagine that a particle is launched from origin with an initial speed $v_i$ at an angle $\theta_i$ above the horizontal.

The components of the particle’s initial velocity are:

$$v_{xi} = v_i \cos \theta_i$$

$$v_{yi} = v_i \sin \theta_i$$  \hspace{1cm} (3)

We shall assume that the particle experiences no air resistance. Therefore:

$$a_x = 0 \quad \text{and} \quad a_y = -g$$  \hspace{1cm} (4)

Substituting these accelerations into our kinematic equations:

$$v_{xf} = v_{xi}$$

$$v_{yf} = v_{yi} - gt$$

Notice that the horizontal and vertical velocities are independent of each other. This means that we can solve this two dimensional motion problem as two separate one dimensional motion problems. The positions of our particle are:
We shall now consider the case where \( y_f = y_i \). For simplicity we will assume that \( x_i = 0 \) and \( y_i = 0 \).

The horizontal distance traveled by the particle is called the **range** \((R)\). The range is:

\[
R = v_{xi}t = v_i \cos \theta_t t
\]

From the above equation, we can see that to compute the range of the particle we must first compute the time of flight. To find this, we look at the vertical motion of the particle:

\[
y_f = y_i + v_i \sin \theta_i t - \frac{1}{2}gt^2
\]

\[
\frac{1}{2}gt^2 = v_i \sin \theta_i t
\]

\[
t = \frac{2v_i \sin \theta_i}{g}
\]
The horizontal range of the particle is:

\[ R = v_i \cos \theta_i t \]

\[ \frac{2v_i^2}{g} \sin \theta_i \cos \theta_i \]

\[ = \frac{v_i^2}{g} \sin 2\theta_i \] (6)

Imagine that we wanted to find the range of our particle at \( \alpha = \frac{\pi}{2} - \theta_i \):

\[ R = \frac{v_i^2}{g} \sin 2\alpha \]

\[ = \frac{v_i^2}{g} \sin \left( \frac{\pi}{2} - \theta_i \right) \]

\[ = \frac{v_i^2}{g} \left[ \sin \pi \cos 2\theta_i - \cos \pi \sin 2\theta_i \right] \]

\[ = \frac{v_i^2}{g} \sin 2\theta_i \]

Therefore the range of the particle is the same for both angles.

\[ R(\theta_i) = R\left( \frac{\pi}{2} - \theta_i \right) \]
Example 2: Wile E. Coyote plans to use his new ACME super cannon to crush the Road Runner. He places the cannon at the edge of a 150 m high cliff. The launch speed of the cannon ball is 20 m/s. How far from the base of the cliff should he put the bird feed?

Solution:
Example 3: During a tennis match, a player serves the ball at 23.6 m/s, with the center of the ball leaving the racquet horizontally 2.37 m above the court surface. The net is 12 m away and 0.90 m high. When the ball reaches the net, (a) does the ball clear it and (b) what is the distance between the center of the ball and the top of the net? Suppose that, instead, the ball is served as before but now it leaves the racquet at 5.00° below the horizontal. When the ball reaches the net, (c) does the ball clear it and (d) what now is the distance between the center of the ball and the top of the net?

Solution:
Example 4: You throw a ball toward a wall at speed 25.0 m/s and at angle $\theta_0 = 40.0^\circ$ above the horizontal. The wall is distance $d = 22.0$ m from the release point of the ball. (a) How far above the release point does the ball hit the wall? What are the (b) horizontal and (c) vertical components of its velocity as it hits the wall? (d) When it hits, has it passed the highest point on its trajectory?

Solution:
Uniform Circular Motion

When an object moves in a circular path at a constant speed, the object’s motion is called **uniform circular motion**. Since the object’s direction is constantly changing, this acceleration, called the **centripetal acceleration**, points toward the center of the circular path and has a magnitude:

\[ a_c = \frac{v^2}{r} \quad (7) \]

From the diagram (b) above, the average acceleration is:

\[ \bar{a}_{\text{avg}} = \frac{\bar{v}_f - \bar{v}_i}{\Delta t} = \frac{\Delta \bar{v}}{\Delta t} = \bar{a} \quad (\text{since acceleration is constant}) \]

The two triangles (b and c) are similar, therefore:

\[ \frac{\Delta \bar{v}}{\bar{v}} = \frac{\Delta \bar{r}}{\bar{r}} \Rightarrow \Delta \bar{v} = \frac{\bar{v}}{\bar{r}} \Delta \bar{r} \]

Combining these two results, our acceleration vector is:

\[ \bar{a} = \lim_{\Delta t \to 0} \frac{v}{r} \left( \frac{\Delta r}{\Delta t} \right) = \frac{v}{r} \frac{dr}{dt} = \frac{v}{r} = \frac{v^2}{r} \]

If we look at the triangle (c), we see that the change in velocity vector points towards the center of the circular path. Therefore the acceleration vector will point toward the center of the circular path.
Curved Paths

When moving along a curved path, the velocity vector changes in direction and magnitude. The particle is moving to the right along a partial circular path. This means that the direction of the acceleration vector changes at every point. *How do we know?* Since the particle’s speed is changing, there is a component of the acceleration vector tangent to the circle. This is called the **tangential acceleration**:

$$a_t = \frac{d|\vec{v}|}{dt} \quad (8)$$

The other component of the acceleration vector is directed along the radius of the curved path. This is called the **radial acceleration**:

$$a_r = a_c = \frac{v^2}{r} \quad (9)$$

Since the radial acceleration always points towards the center of the curved path and the tangential acceleration always points along the direction of motion, we know that the direction of the acceleration vector is always changing. The magnitude of our acceleration vector is:

$$|\vec{a}| = \sqrt{a_t^2 + a_r^2}$$
Example 5: A boy whirls a string in a horizontal circle of radius 0.8 m. How many revolutions per minute does the ball make if the magnitude of the centripetal acceleration is g?

Solution:
**Example 6:** An automobile whose speed is increasing at a rate of \(0.850 \frac{m}{s^2}\) travels along a circular road of radius 15.0 m. When the instantaneous speed of the automobile is 15.00 m/s, find (a) the tangential acceleration component, (b) the radial acceleration component, and (c) the magnitude and direction of the total acceleration.

**Solution:**
Relative Motion

We are now going to describe how different observations may be made by observers in different frames of reference. Imagine that you are in an airplane moving at 500 mi/hr toward the east, then your velocity would also be 500 mi/hr to the east. However, your velocity might be measured with respect to the Earth or to the wind which is trying to blow you off course. What is your speed relative to the airplane?

Imagine that a particle P moves with velocity \( \vec{v}_{P0} \) as measured by an observer O in a frame S. An observer O’ in S’ observes that the velocity of the particle is \( \vec{v}_{P0}' \). The observer O’ moves at a velocity \( \vec{v}_{O'O} \) with respect to O. These relative velocities are related to each other by:

\[
\vec{v}_{P0}' = \vec{v}_{P0} - \vec{v}_{O'O}
\]

(10)

Example 7: A plane flies at an airspeed of 250 km/h. There is a wind blowing at 80 km/h in the northeast direction at exactly 45° to the east of north.

a. In what direction should the plane head in order to fly due north?
b. What is the speed of the plane relative to the ground?

Solution:
Example 8: Two boat landings are 2.0 km apart on the same bank of a stream that flows at 1.4 km/h. A motor boat makes the round trip between the two landings in 50 min. What is the speed of the boat relative to the water?

Solution: