1. Today 1 euro is worth 1.25 dollars. If the value of the euro in dollars increases by 10% tomorrow, approximately how many euros will 2.20 dollars be worth then?

A. 1.50 B. 1.58 C. 1.60 D. 1.76 E. 1.94

2. The lines with equations 2x - y = a and y - x = b intersect at the point (p, q). Find the value of q.

A. a + b B. a - b C. 2a + b D. a + 2b E. 2a - b

3. Find $\log_{10}\left(\log_{10}\left(\log_{10} 10^{10000000000}\right)\right)$. A. 0 B. 1 C. 2 D. 3 E. 6

4. The digits of a number are rearranged, and the resulting number is added to the original number. How many of the numbers below could NOT equal this sum?
777 7,777 77,777 777,777 7,777,777

A. 0 B. 1 C. 2 D. 3 E. 4

5. Perpendicular lines L and M have equations Ax + By = D and Cx + Ay = E, respectively (A·B ≠ 0). If the sum of these equations is 6x + 10y = 12, one of the lines must have slope

A. -2 B. $-\frac{1}{2}$ C. $-\frac{1}{4}$ D. $\frac{1}{4}$ E. 4

6. In the equation AMA - TYC = SML, identical letters are replaced by the same digit 0 to 9, and different letters are replaced by different digits 0 to 9. If A = 4, which of the following is a possible value of M?

A. 1 B. 3 C. 6 D. 8 E. 9

7. In a sample of 5 positive data values, the median, minimum, and range are all equal, and the mean equals one of the values. The ratio of the maximum to the mean is

A. 1.6 B. 1.75 C. 1.8 D. 2 E. 2.4

8. The points (6, 4) and (2, 10) are symmetric with respect to the line L. An equation for line L is

A. 2x - 3y = 13 B. 3x + 2y = 26 C. 2x + 3y = 29 D. 3y - 2x = 13 E. 2y - 3x = 2

9. The solution to the equation $\left(\log_8 x^2\right)\left(\log_x 8\right)^2 = 1$ satisfies which inequality below?

A. $0 < x \le 1$ B. $1 < x \le 10$ C. $10 < x \le 50$ D. $50 < x \le 100$ E. $x > 100$

10. Knaves always lie; knights always tell the truth. Al says, “Bo is a knight,” Bo says, “Cy is a knave,” and Cy says, “Exactly one of Al and Bo is a knave”. If Al, Bo, and Cy are each either a knight or a knave, it is true that

A. Al and Cy are both knights B. Al and Cy are both knaves
C. Al is a knight, Cy is a knave D. Al is a knave, Cy is a knight
E. it cannot be determined what Al and Cy are

11. The equation $a^4 + 2b^2 + c^2 = 2013$ has a unique solution in positive integers. For this solution, find $a + b + c$. A. 36 B. 37 C. 38 D. 39 E. 40

12. In the sequence $\{a_n\}$, $a_n = \dfrac{a_1 + a_2 + \ldots + a_{n-1}}{n-1}$ for n ≥ 3. If $a_1 + a_2 \neq 0$ and the sum of the first N terms is $12(a_1 + a_2)$, find N. A. 16 B. 18 C. 20 D. 22 E. 24

13. If S = {(*x*, *y*): *x*, *y* are integers and $x^2 = 4y^2 + 81$}, how many elements are in S?

A. 2 B. 4 C. 6 D. 8 E. 10

14. Find the value of *k* for which the equation $\left|k - \left||x| - 6\right|\right| = 2$ has exactly 5 solutions. Write your answer in the corresponding blank on the answer sheet.

15. All fractions 0 < $\dfrac{a}{b}$ < 1 (*a*, *b* positive integers) are placed into the sequence $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \cdots$ first by increasing order of denominator and then by increasing order of numerator. Find *a* + *b* for the 2013th element of the sequence.

A. 124 B. 125 C. 126 D. 127 E. 128

16. In parallelogram *ABCD*, $\overline{BC}$ is extended beyond point *C* to point *E*. Points *F* and *G* are the points of intersection of $\overline{AE}$ with $\overline{BD}$ and $\overline{CD}$, respectively. If *FG* = 12 and *EG* = 15, find *AF*.

A. 16 B. 18 C. 20 D. 24 E. 27

17. Ha and Mo play the following game: a fair coin is flipped repeatedly. Ha chooses a 3-outcome sequence, and then Mo chooses a different 3-outcome sequence. Whoever's sequence occurs first wins. If Ha chooses HHH, which choice gives Mo the greatest probability of winning?

A. THH B. THT C. TTH D. HTT E. TTT

18. If Mo chooses the optimal sequence in Problem 17, the probability that Mo wins is

A. $\frac{3}{5}$ B. $\frac{5}{8}$ C. $\frac{3}{4}$ D. $\frac{4}{5}$ E. $\frac{7}{8}$

19. All of the coefficients of the fourth degree polynomial P(x) are odd integers. Find the maximum possible number of rational solutions of the equation P(x) = 0.

A. 0 B. 1 C. 2 D. 3 E. 4

20. In rectangle ABCD, point E lies between A and B and point F lies between B and C. The areas of ΔADE, ΔEBF, and ΔDCF are all equal. If AB = 4 and BC = 2, find the ratio of the area of ΔDEF to the area of ΔADE.

A. $\frac{4\sqrt{3}}{3}$ B. $\sqrt{5}$ C. $2\sqrt{2}$ D. $\frac{2\sqrt{10}}{3}$ E. $\sqrt{6}$

Test #1 **AMATYC Student Mathematics League** October/November 2013

1. C
2. D
3. B
4. B
5. D
6. B
7. A
8. D
9. D
10. C
11. B
12. E
13. E
14. 8
15. A
16. B
17. A
18. E
19. A
20. B

1. A total of 50 problems, minus 12 problems in common, makes 38 distinct problems in all. (Answer: D)

2. The third side of a triangle must be longer than the difference of the other two sides and shorter than their sum. Therefore if c is the length of the third side: $8.1 - 1.4 < c < 8.1 + 1.4 \implies 6.7 < c < 9.5$. Of the choices provided, 8 is the only number that falls into this range. (Answer: D)

3. The first equation minus the second is $(3e)x + (3e)y = 3e \implies x + y = 1 \implies x = 1 - y$. Substitute for x in the first equation to get $y = 2 \implies x = -1 \implies b - a = y - x = 3$. (Answer: E)

4. Just factor the numbers given: $2014 = 2 \cdot 19 \cdot 53 \implies \{1, 18, 52\}$, $2015 = 5 \cdot 13 \cdot 31 \implies \{4, 12, 30\}$, and $2016 = 2^5 \cdot 3^2 \cdot 7 \implies \{1, 2, 6\}$, which has the desired property. (Answer: C)

5. The lines intersect at some point $(x, 0)$. Set $y = 0$ in each equation to find $x = -b/2$ and $x = 6/m$, respectively. These ae the same point, so $-b/2 = 6/m \implies mb = -12$. (Answer: B)

6. Play around a bit, starting with $n = 3$, and hopefully find $\frac{1}{4} = \frac{1}{20} + \frac{1}{5} = \frac{1}{12} + \frac{1}{6} = \frac{1}{8} + \frac{1}{8}$. In general, if $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{a+b}{ab} = \dfrac{1}{n} = \dfrac{k}{kn}$ with k *as small as possible*, then $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{1}{kn} + \dfrac{k-1}{kn}$, so $k-1$ is a factor of n, and the pairs (a, b) with $a \geq b > 0$ and $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{1}{n}$ are of the form $(a, b) = (kn, kn/(k-1))$, where $k - 1$ is a factor of n: $k = 2$ is the smallest possible, corresponding to $\frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$, and $k = n + 1$ is the largest possible, corresponding to $\frac{1}{n(n+1)} + \frac{1}{n+1} = \frac{1}{n}$. Therefore, the number of solutions is the number of factors of n, and the smallest n with 3 factors is $n = 4$. (Answer: B)

7. The possibilities for b are fewest, so with a calculator, store the values $5, 10, \ldots$ for B, and use the TABLE feature with formula $Y = \sqrt{2013 - B^2 - X^3}$ to find integer pairs $(a, c) = (Y, X)$. The solution $(a, b, c) = (4, 10, 43)$ is quickly found this way, so $a + b + c = 57$. (Answer: B)

8. $A = 11$, since otherwise two different letters are both 11 or some letter is $\geq 33 > 27$. From $MTYC = 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7$, similar considerations demand that some letter is $3 \cdot 5 = 15$ and the others are 3, 5, and 7, so $M + T + Y + C = 3 + 5 + 7 + 15 = 30$. (Answer: A)

9. As sets of values, $\{P(0), P(3)\} = \{1, 139\}$ and $\{P(1), P(2)\} = \{1, 689\}$ or $\{13, 53\}$. The coefficients of P are non-negative, so P is increasing on $[0, \infty)$, and the values must be $P(0) = 1, P(1) = 13, P(2) = 53, P(3) = 139$. One way to continue is to set $P(x) = ax^3 + bx^2 + cx + d$, use the above values to write the equations $P(0) = d = 1, P(1) = a + b + c + d = 13, P(2) = 8a + 4b + 2c + d = 53$, and $P(3) = 27a + 9b + 3c + d = 139$, and solve these to find $(a, b, c, d) = (3, 5, 4, 1) \implies P(-1) = -a+b-c+d = -3+5-4+1 = -1$. Alternatively, if you know that the k^{th} differences of a k^{th}-degree polynomial are constant, you can use this fact to quickly find the same result. (Answer: B)

10. Let $\cos_{RAD}(x)$ be the cosine function which takes a radian argument, and let $\cos_{DEG}(x)$ be the cosine function which takes a degree argument. The relation between these is $\cos_{DEG}(x) = \cos_{RAD}(\pi x/180)$, so the problem is to find the smallest positive solution to $\cos_{RAD}(x) = \cos_{DEG}(x) \iff \cos_{RAD}(x) = \cos_{RAD}(\pi x/180)$. With a graphing calculator (in radian mode), it is easy to find that the first positive intersection of the curves $Y_1 = \cos(X)$ and $Y_2 = \cos(\pi X/180)$ occurs at approximately $(6.1754042, 0.99419723)$. (Answer: 6.175)

11. By the Pythagorean theorem, $BD = 10$, so $\triangle ABD$ is isoceles with base $AB = 6$ and sides $BD = DA = 10$. Let $2\alpha = \angle A$; by the law of sines, $\frac{BE}{\sin\alpha} = \frac{6}{\sin(\pi - 3\alpha)} = \frac{6}{\sin 3\alpha}$ and $\frac{10-BE}{\sin\alpha} = \frac{10}{\sin 3\alpha}$. It follows that $\frac{\sin\alpha}{\sin 3\alpha} = \frac{BE}{6} = \frac{10-BE}{10} \implies 10BE = 60 - 6BE \implies BE = \frac{15}{4}$ (Answer: A)

12. There are two possibilites each for L and M, so 4 possible points of intersection: $(a, b) = (0, 4), (4, 0), (-4, 12)$ or $(12, -4) \implies 3a + b = 4, 12, 0$, or 32, so only 8 is not possible. (Answer: C)

13. If n = length of the first trip and k = number of trips, then $n+(n+2)+(n+4)+\cdots+(n+2(k-1)) = 366 \implies kn + 2(1+2+\cdots+(k-1)) = k(n+k-1) = 366 = 2\cdot 3\cdot 61$. k must be a factor of 366, so the positive integer solutions are $(k,n) = (1,366), (2,182), (3,120)$, and $(6,56)$; since $n \leq 90$, only the last of these works, and the trips were of lengths 56, 58, 60, 62, 64, and 66. (Answer: B)

14. Most likely, the problem should have been: "For a 6-digit bit string s, let $R(s)$ be the reverse of s and let $O(s) = 111111 - s$ be the opposite of s; e.g., $R(110101) = 101011$ and $O(110101) = 001010$. Find the largest possible size of a set S of 6-digit bit strings, such that $s \in S \implies R(s), O(s) \notin S$." Each of the $2^6 = 64$ strings is either a palindrome, with $R(s) = s$; a palopposite, with $R(s) = O(s)$; or neither. There are $2^3 = 8$ palindromes, and none may be in S. There are $2^3 = 8$ palopposites which form pairs such as $\{011001, 100110\}$, and at most one from each of these 4 pairs may be in S. The remaining 48 strings fall into 12 quartets of the form $\{s, R(s), O(s), R(O(s)) = O(R(s))\}$; at most 2 from this quartet may be in S, either $\{s, O(R(s))\}$ or $\{R(s), O(s)\}$. Thus, S contains at most $4 + 2(12) = 28$ strings. S is not unique – there are 2^{16} such sets! Writing strings as decimal numbers, one example is $S = \{7, 11, 21, 25, 1, 31, 2, 47, 3, 15, 4, 55, 5, 23, 6, 39, 9, 27, 10, 43, 13, 19, 14, 35, 17, 29, 22, 37\}$. So the correct answer to the likely problem is 28, which was not an option. (Answer: Correct for all students)

15. The non-intersecting "diagonals" PR and QS lie on perpendicular lines (which intersect at T), so the area is $\frac{1}{2}|PR||QS|$. $\triangle QTS \cong \triangle PTR$, so $|QS| = |PR| = 8\sqrt{2}$, so the area is exactly 64. (Answer: E)

16. In other words, find the smallest pair (a,b) with $a^2 = 2b^2+2$ and $a > 10$. Use the TABLE function on a calculator with $Y = \sqrt{2X^2+2}$ to quickly find the pair $(a,b) = (58,41)$, so $a - b = 17$. (Answer: C)

17. Just write out the possibilities to find 2 such numbers that begin with 1 (13524, 14253), 3 that begin with 2 (24135, 24153, 25314), and 4 that begin with 3 (31425, 31524, 35241, 35142); by symmetry, there are 3 that begin with 4 and 2 that begin with 5, so 14 such numbers with no consecutive digits. There are 5! 5-digit numbers with distinct digits, so the probability is $14/5! = 7/60$. (Answer: B)

18. The region is the union of a quarter-circle C_4 of radius 4 in the first quadrant, a quarter-circle C_3 of radius 3 in the second quadrant, and the triangle T with vertices $O(0,0), P(-3,0), Q(0,4)$. Estimate the area inside T but outside C_3 by a right triangle with height 1 and base 3/4, to find $A > \frac{\pi}{4}(3^2+4^2) + \frac{1}{2}(1)(\frac{3}{4}) \approx 20.009954$; the neglected area is contained in the right triangle with vertices $(0,3), (0,\sqrt{8})$, and $(-1,\sqrt{8})$, which has area $(3-\sqrt{8})/2 \approx 0.0858$, so $20.009 < A < 20.096$, so only B works. Alternatively, solve $y = \frac{4}{3}x + 4$ and $x^2 + y^2 = 9$ to find that T and C_3 intersect at $R(-21/25, 72/25)$. The area of $\triangle OQR$ is 42/25 and the area of the remaining sector of C_3 is $\frac{1}{2}3^2\arctan(72/21)$, so the exact area is $4\pi + 4.5\arctan(24/7) + 42/25 \approx 20.03788 \approx 20.04$. (Answer: B)

19. By the quadratic formula, these polynomials factor iff $m^2 - 4n$ and $m^2 + 4n$ are perfect squares. If your calculator can deal with two-variable tables, look for integer values of $\sqrt{m^2-4n} + \sqrt{m^2+4n}$, $1 \leq m, n \leq 99$. Otherwise, suppose $m^2 - 4n = (m-k)^2$ and $m^2 + 4n = (m+j)^2$ for some integers $j, k > 0$; it follows that $2mj + j^2 = 4n = 2mk - k^2 \implies 2m(k-j) = j^2 + k^2 \implies j, k$ are both odd or both even $\implies$ both sides are divisible by 4 $\implies j = 2p$ and $k = 2q$ for some $p, q \geq 1 \implies m(q-p) = p^2 + q^2 \implies q = p + d$ for some $d \geq 1 \implies m = \frac{2p^2}{d} + 2p + d$ and $n = (2mj + j^2)/4 = p(m+p)$. Plug in values of p and d for which $d|2p^2$ and record those pairs with $m, n < 100$; since $m > 2p \implies n > 3p^2$, it is only necessary to check through $p = 5$ and, writing (m,n) instead of (n,m) as on the test, find the 7 pairs $(m,n) = (5,6), (13,30), (10,24), (25,84), (17,60), (15,54)$, and $(20,96)$. (Answer: D)

20. The triangles have right angles at A and B, so $BC = \sqrt{50^2 - 40^2} = 30$, $area(\triangle BCD) = \frac{1}{2}(30)(40) = 600$, $AC = \sqrt{50^2 - 14^2} = 48$, and $area(\triangle ACD) = \frac{1}{2}(14)(48) = 336$. In coordinates with C at the origin, D at $(50,0)$, and A to the right of B, CB is on the line $y = 7x/24$ and BD is on the line $y = 3(50-x)/4$, so the lines intersect at $E = (36, 21/2)$. Therefore, $area(\triangle ACD \cup \triangle BCD) = area(\triangle ACD) + area(\triangle BCD) - area(\triangle ECD) = 600 + 336 - \frac{1}{2}(50)(21/2) = 673.5$. (Answer: D)