1. Ms. Pham writes 2 final exams, each with 25 problems. If the exams have 12 problems in common, how many problems does she write?

   A. 24 B. 26 C. 37 D. 38 E. 49

2. A triangle has two sides of length 8.1 and 1.4. If the length of the third side is an even integer, its length must be

   A. 2 B. 4 C. 6 D. 8 E. 10

3. If (a, b) is the solution to the system of equations $\begin{cases} \pi x + (\pi + e)y = \pi + 2e \\ (\pi + 3e)x + (\pi + 4e)y = \pi + 5e \end{cases}$, find b - a.

   A. -3 B. -1 C. 0 D. 1 E. 3

4. The year 2013 has the property that when its distinct prime factors 3, 11, and 61 are each reduced by 1 and written in increasing order (that is, 2, 10, 60) each number is a factor of the next. Find the next year with this property.

   A. 2014 B. 2015 C. 2016 D. 2017 E. 2018

5. If the lines with equations y = 2x + b and y = mx - 6 intersect at a point on the x-axis, then A. mb = 12 B. mb + 12 = 0 C. m = 3b D. m + 3b = 0 E. 3m = b

6. Find the smallest positive integer value of n for which $\frac{1}{a} + \frac{1}{b} = \frac{1}{n}$ has at least three solutions (a, b) in integers with $a \geq b > 0$.

   A. 3 B. 4 C. 5 D. 6 E. 8

7. The equation $a^3 + b^2 + c^2 = 2013$ has a solution in positive integers for which b is a multiple of 5. Find a + b + c for this solution.

   A. 55 B. 57 C. 59 D. 61 E. 63

8. Each letter A through Z of the alphabet is assigned a unique integer from 2 to 27. If $A \cdot M \cdot A \cdot T \cdot Y \cdot C = 3^2 \cdot 5^2 \cdot 7 \cdot 11^2$, find M + T + Y + C.

   A. 30 B. 34 C. 36 D. 38 E. 42

9. The third-degree polynomial P(x) has only nonnegative integer coefficients. If $P(0) \cdot P(3) = 139$ and $P(1) \cdot P(2) = 689$, find P(-1).

   A. -2 B. -1 C. 0 D. 1 E. 2

10. Find the smallest positive value of t such that cos t is the same whether t is in radians or in degrees. Write your answer (rounded to 3 decimal places) in the corresponding blank on the answer sheet.

11. In quadrilateral ABCD, AB = 6, BC = 6, CD = 8, AD = 10, and $\angle C = 90°$. If the angle bisector of $\angle A$ meets diagonal BD at point E, find BE.

    A. $\frac{15}{4}$ B. 4 C. 5 D. 6 E. $\frac{25}{4}$

12. Line L has intercepts 2 and 4, while line M has intercepts 4 and 6. If L and M intersect at (a, b), which of the following could NOT be 3a + b?

A. 0 B. 4 C. 8 D. 12 E. 32

13. Sue traveled continuously starting on 1/1/2012. Her first trip was less than 3 months, and each successive trip was 2 days longer than the previous trip. If her last trip ended on 12/31/2012, which of these was the length in days of one of her trips? A. 54 B. 58 C. 65 D. 72 E. 77

14. A binary string is a sequence of 1's and 0's, such as 10011 or 11101010. How many different binary strings of length 6 are there such that no two are reversals of each other or add up to 111111?

A. 22 B. 23 C. 24 D. 25 E. 26

15. In quadrilateral PQRS, $\angle P = \angle Q = \angle S = 45°$, $\angle QPR = \angle RPS$, and $PR = 8\sqrt{2}$. Find the area of quadrilateral PQRS to the nearest integer.

A. 60 B. 61 C. 62 D. 63 E. 64

16. The numbers 2 and 1 are the smallest positive integers for which the square of the first is 2 more than twice the square of the second. If a and b are the smallest such pair with a > 10, find a - b.

A. 13 B. 15 C. 17 D. 19 E. 21

17. A number is chosen at random from among all 5-digit numbers containing exactly one each of the digits 1, 2, 3, 4, and 5. Find the probability that no two adjacent digits in the number are consecutive integers.

A. $\frac{1}{10}$ B. $\frac{7}{60}$ C. $\frac{2}{15}$ D. $\frac{3}{20}$ E. $\frac{1}{6}$

18. The triangular region with vertices (0, 0), (4, 0), and (0, 3) is rotated 90° counter-clockwise around the origin. Find the area of the figure formed by this rotation to the nearest hundredth.

A. 19.96 B. 20.04 C. 20.12 D. 20.20 E. 20.28

19. For how many pairs of positive integers (n, m) with n, m < 100 are both of the polynomials $x^2 + mx + n$ and $x^2 + mx - n$ factorable over the integers?

A. 4 B. 5 C. 6 D. 7 E. 8

20. Triangles ACD and BCD (AD = 14, BD = 40) are inscribed in a semicircle with diameter CD = 50. If AB > 25, find the area of their union.

A. 625 B. 637.5 C. 652.5 D. 673.5 E. 675

1. D
2. D
3. E
4. C
5. B
6. B
7. B
8. A
9. B
10. 6.175
11. A
12. C
13. B
14. Correct for all students
15. E
16. C
17. B
18. B
19. D
20. D

1. A total of 50 problems, minus 12 problems in common, makes 38 distinct problems in all. (Answer: D)

2. The third side of a triangle must be longer than the difference of the other two sides and shorter than their sum. Therefore if c is the length of the third side: $8.1-1.4 < c < 8.1+1.4 \implies 6.7 < c < 9.5$. Of the choices provided, 8 is the only number that falls into this range. (Answer: D)

3. The first equation minus the second is $(3e)x+(3e)y = 3e \implies x+y = 1 \implies x = 1-y$. Substitute for x in the first equation to get $y = 2 \implies x = -1 \implies b-a = y-x = 3$. (Answer: E)

4. Just factor the numbers given: $2014 = 2\cdot 19\cdot 53 \implies \{1, 18, 52\}$, $2015 = 5\cdot 13\cdot 31 \implies \{4, 12, 30\}$, and $2016 = 2^5\cdot 3^2\cdot 7 \implies \{1, 2, 6\}$, which has the desired property. (Answer: C)

5. The lines intersect at some point $(x, 0)$. Set $y = 0$ in each equation to find $x = -b/2$ and $x = 6/m$, respectively. These ae the same point, so $-b/2 = 6/m \implies mb = -12$. (Answer: B)

6. Play around a bit, starting with $n = 3$, and hopefully find $\frac{1}{4} = \frac{1}{20}+\frac{1}{5} = \frac{1}{12}+\frac{1}{6} = \frac{1}{8}+\frac{1}{8}$. In general, if $\dfrac{1}{a}+\dfrac{1}{b} = \dfrac{a+b}{ab} = \dfrac{1}{n} = \dfrac{k}{kn}$ with k *as small as possible*, then $\dfrac{1}{a}+\dfrac{1}{b} = \dfrac{1}{kn}+\dfrac{k-1}{kn}$, so $k-1$ is a factor of n, and the pairs (a, b) with $a \geq b > 0$ and $\dfrac{1}{a}+\dfrac{1}{b} = \dfrac{1}{n}$ are of the form $(a, b) = (kn, kn/(k-1))$, where $k-1$ is a factor of n: $k = 2$ is the smallest possible, corresponding to $\frac{1}{2n}+\frac{1}{2n} = \frac{1}{n}$, and $k = n+1$ is the largest possible, corresponding to $\frac{1}{n(n+1)}+\frac{1}{n+1} = \frac{1}{n}$. Therefore, the number of solutions is the number of factors of n, and the smallest n with 3 factors is $n = 4$. (Answer: B)

7. The possibilities for b are fewest, so with a calculator, store the values $5, 10, \ldots$ for B, and use the TABLE feature with formula $Y = \sqrt{2013-B^2-X^3}$ to find integer pairs $(a, c) = (Y, X)$. The solution $(a, b, c) = (4, 10, 43)$ is quickly found this way, so $a+b+c = 57$. (Answer: B)

8. $A = 11$, since otherwise two different letters are both 11 or some letter is $\geq 33 > 27$. From $MTYC = 3\cdot 3\cdot 5\cdot 5\cdot 7$, similar considerations demand that some letter is $3\cdot 5 = 15$ and the others are 3, 5, and 7, so $M+T+Y+C = 3+5+7+15 = 30$. (Answer: A)

9. As sets of values, $\{P(0), P(3)\} = \{1, 139\}$ and $\{P(1), P(2)\} = \{1, 689\}$ or $\{13, 53\}$. The coefficients of P are non-negative, so P is increasing on $[0, \infty)$, and the values must be $P(0) = 1, P(1) = 13, P(2) = 53, P(3) = 139$. One way to continue is to set $P(x) = ax^3+bx^2+cx+d$, use the above values to write the equations $P(0) = d = 1, P(1) = a+b+c+d = 13, P(2) = 8a+4b+2c+d = 53$, and $P(3) = 27a+9b+3c+d = 139$, and solve these to find $(a, b, c, d) = (3, 5, 4, 1) \implies P(-1) = -a+b-c+d = -3+5-4+1 = -1$. Alternatively, if you know that the k^{th} differences of a k^{th}-degree polynomial are constant, you can use this fact to quickly find the same result. (Answer: B)

10. Let $\cos_{RAD}(x)$ be the cosine function which takes a radian argument, and let $\cos_{DEG}(x)$ be the cosine function which takes a degree argument. The relation between these is $\cos_{DEG}(x) = \cos_{RAD}(\pi x/180)$, so the problem is to find the smallest positive solution to $\cos_{RAD}(x) = \cos_{DEG}(x) \iff \cos_{RAD}(x) = \cos_{RAD}(\pi x/180)$. With a graphing calculator (in radian mode), it is easy to find that the first positive intersection of the curves $Y_1 = \cos(X)$ and $Y_2 = \cos(\pi X/180)$ occurs at approximately $(6.1754042, 0.99419723)$. (Answer: 6.175)

11. By the Pythagorean theorem, $BD = 10$, so $\triangle ABD$ is isoceles with base $AB = 6$ and sides $BD = DA = 10$. Let $2\alpha = \angle A$; by the law of sines, $\frac{BE}{\sin\alpha} = \frac{6}{\sin(\pi-3\alpha)} = \frac{6}{\sin 3\alpha}$ and $\frac{10-BE}{\sin\alpha} = \frac{10}{\sin 3\alpha}$. It follows that $\frac{\sin\alpha}{\sin 3\alpha} = \frac{BE}{6} = \frac{10-BE}{10} \implies 10BE = 60-6BE \implies BE = \frac{15}{4}$ (Answer: A)

12. There are two possibilites each for L and M, so 4 possible points of intersection: $(a, b) = (0, 4), (4, 0), (-4, 12)$ or $(12, -4) \implies 3a+b = 4, 12, 0$, or 32, so only 8 is not possible. (Answer: C)

13. If n = length of the first trip and k = number of trips, then $n+(n+2)+(n+4)+\cdots+(n+2(k-1)) = 366 \implies kn + 2(1+2+\cdots+(k-1)) = k(n+k-1) = 366 = 2\cdot 3\cdot 61$. k must be a factor of 366, so the positive integer solutions are $(k,n) = (1,366), (2,182), (3,120)$, and $(6,56)$; since $n \le 90$, only the last of these works, and the trips were of lengths 56, 58, 60, 62, 64, and 66. (Answer: B)

14. Most likely, the problem should have been: "For a 6-digit bit string s, let $R(s)$ be the reverse of s and let $O(s) = 111111 - s$ be the opposite of s; e.g., $R(110101) = 101011$ and $O(110101) = 001010$. Find the largest possible size of a set S of 6-digit bit strings, such that $s \in S \implies R(s), O(s) \notin S$." Each of the $2^6 = 64$ strings is either a palindrome, with $R(s) = s$; a palopposite, with $R(s) = O(s)$; or neither. There are $2^3 = 8$ palindromes, and none may be in S. There are $2^3 = 8$ palopposites which form pairs such as $\{011001, 100110\}$, and at most one from each of these 4 pairs may be in S. The remaining 48 strings fall into 12 quartets of the form $\{s, R(s), O(s), R(O(s)) = O(R(s))\}$; at most 2 from this quartet may be in S, either $\{s, O(R(s))\}$ or $\{R(s), O(s)\}$. Thus, S contains at most $4 + 2(12) = 28$ strings. S is not unique – there are 2^{16} such sets! Writing strings as decimal numbers, one example is $S = \{7, 11, 21, 25, 1, 31, 2, 47, 3, 15, 4, 55, 5, 23, 6, 39, 9, 27, 10, 43, 13, 19, 14, 35, 17, 29, 22, 37\}$. So the correct answer to the likely problem is 28, which was not an option. (Answer: Correct for all students)

15. The non-intersecting "diagonals" PR and QS lie on perpendicular lines (which intersect at T), so the area is $\frac{1}{2}|PR||QS|$. $\triangle QTS \cong \triangle PTR$, so $|QS| = |PR| = 8\sqrt{2}$, so the area is exactly 64. (Answer: E)

16. In other words, find the smallest pair (a,b) with $a^2 = 2b^2+2$ and $a > 10$. Use the TABLE function on a calculator with $Y = \sqrt{2X^2+2}$ to quickly find the pair $(a,b) = (58,41)$, so $a - b = 17$. (Answer: C)

17. Just write out the possibilities to find 2 such numbers that begin with 1 (13524, 14253), 3 that begin with 2 (24135, 24153, 25314), and 4 that begin with 3 (31425, 31524, 35241, 35142); by symmetry, there are 3 that begin with 4 and 2 that begin with 5, so 14 such numbers with no consecutive digits. There are 5! 5-digit numbers with distinct digits, so the probability is $14/5! = 7/60$. (Answer: B)

18. The region is the union of a quarter-circle C_4 of radius 4 in the first quadrant, a quarter-circle C_3 of radius 3 in the second quadrant, and the triangle T with vertices $O(0,0), P(-3,0), Q(0,4)$. Estimate the area inside T but outside C_3 by a right triangle with height 1 and base 3/4, to find $A > \frac{\pi}{4}(3^2+4^2) + \frac{1}{2}(1)(\frac{3}{4}) \approx 20.009954$; the neglected area is contained in the right triangle with vertices $(0,3), (0,\sqrt{8})$, and $(-1,\sqrt{8})$, which has area $(3-\sqrt{8})/2 \approx 0.0858$, so $20.009 < A < 20.096$, so only B works. Alternatively, solve $y = \frac{4}{3}x + 4$ and $x^2 + y^2 = 9$ to find that T and C_3 intersect at $R(-21/25, 72/25)$. The area of $\triangle OQR$ is 42/25 and the area of the remaining sector of C_3 is $\frac{1}{2}3^2\arctan(72/21)$, so the exact area is $4\pi + 4.5\arctan(24/7) + 42/25 \approx 20.03788 \approx 20.04$. (Answer: B)

19. By the quadratic formula, these polynomials factor iff $m^2 - 4n$ and $m^2 + 4n$ are perfect squares. If your calculator can deal with two-variable tables, look for integer values of $\sqrt{m^2-4n} + \sqrt{m^2+4n}$, $1 \le m, n \le 99$. Otherwise, suppose $m^2 - 4n = (m-k)^2$ and $m^2 + 4n = (m+j)^2$ for some integers $j, k > 0$; it follows that $2mj + j^2 = 4n = 2mk - k^2 \implies 2m(k-j) = j^2 + k^2 \implies j, k$ are both odd or both even $\implies$ both sides are divisible by 4 $\implies j = 2p$ and $k = 2q$ for some $p, q \ge 1 \implies m(q-p) = p^2 + q^2 \implies q = p + d$ for some $d \ge 1 \implies m = \frac{2p^2}{d} + 2p + d$ and $n = (2mj + j^2)/4 = p(m+p)$. Plug in values of p and d for which $d|2p^2$ and record those pairs with $m, n < 100$; since $m > 2p \implies n > 3p^2$, it is only necessary to check through $p = 5$ and, writing (m,n) instead of (n,m) as on the test, find the 7 pairs $(m,n) = (5,6), (13,30), (10,24), (25,84), (17,60), (15,54)$, and $(20,96)$. (Answer: D)

20. The triangles have right angles at A and B, so $BC = \sqrt{50^2 - 40^2} = 30$, $area(\triangle BCD) = \frac{1}{2}(30)(40) = 600$, $AC = \sqrt{50^2 - 14^2} = 48$, and $area(\triangle ACD) = \frac{1}{2}(14)(48) = 336$. In coordinates with C at the origin, D at $(50,0)$, and A to the right of B, CB is on the line $y = 7x/24$ and BD is on the line $y = 3(50-x)/4$, so the lines intersect at $E = (36, 21/2)$. Therefore, $area(\triangle ACD \cup \triangle BCD) = area(\triangle ACD) + area(\triangle BCD) - area(\triangle ECD) = 600 + 336 - \frac{1}{2}(50)(21/2) = 673.5$. (Answer: D)