Math 185 (150B) name: ____________
Sec. 7.3 Trigonometric Substitution date: ____________

1. Determine $\displaystyle\int \frac{1}{\sqrt{7+5t^2}}\,dt$.

Here, the form of the radicand reminds us of a^2+u^2, where $a=\sqrt{7}$ and $u=\sqrt{5}\,t$, so we'll let $\sqrt{5}\,t=\sqrt{7}\tan(\theta)$

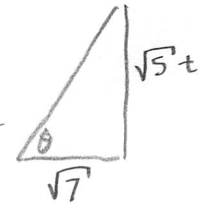
or $t=\frac{\sqrt{7}}{\sqrt{5}}\tan(\theta)$.

Then $dt=\frac{\sqrt{7}}{\sqrt{5}}\sec^2(\theta)\,d\theta$

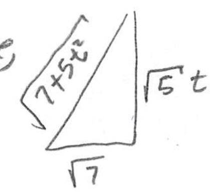
and
$$\sqrt{7+5t^2}=\sqrt{7+5\left(\frac{\sqrt{7}}{\sqrt{5}}\tan(\theta)\right)^2}$$
$$=\sqrt{7+5\cdot\frac{7}{5}\tan^2(\theta)}$$
$$=\sqrt{7+7\tan^2(\theta)}$$
$$=\sqrt{7(1+\tan^2(\theta))}$$
$$=\sqrt{7}\sqrt{1+\tan^2(\theta)}$$
$$=\sqrt{7}\sqrt{\sec^2(\theta)}$$
$$=\sqrt{7}\sec(\theta),\ -\frac{\pi}{2}<\theta<\frac{\pi}{2}$$

So
$$\int\frac{1}{\sqrt{7+5t^2}}\,dt=\int\frac{1}{\sqrt{7}\sec(\theta)}\cdot\frac{\sqrt{7}}{\sqrt{5}}\sec^2(\theta)\,d\theta$$
$$=\int\frac{1}{\sqrt{5}}\sec(\theta)\,d\theta$$
$$=\frac{1}{\sqrt{5}}\int\sec(\theta)\,d\theta$$
$$=\frac{1}{\sqrt{5}}\ln\left|\sec(\theta)+\tan(\theta)\right|+C$$

From our work above, we have
$$t=\frac{\sqrt{7}}{\sqrt{5}}\tan(\theta)$$
so
$$\tan(\theta)=\frac{\sqrt{5}}{\sqrt{7}}\,t$$
but we don't have an expression for $\sec(\theta)$ in terms of t. We'll use $\tan(\theta)=\frac{\sqrt{5}}{\sqrt{7}}\,t$ and some basic trigonometry. Consider the triangle shown at right. From the Pythagorean Theorem, we find the hypotenuse is $\sqrt{7+5t^2}$.



From the second triangle,
$$\cos(\theta)=\frac{\sqrt{7}}{\sqrt{7+5t^2}}$$



so
$$\sec(\theta)=\frac{\sqrt{7+5t^2}}{\sqrt{7}}$$

Finally,
$$\int\frac{1}{\sqrt{7+5t^2}}\,dt=\frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{7+5t^2}}{\sqrt{7}}+\frac{\sqrt{5}}{\sqrt{7}}\,t\right|+C$$
$$=\frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{7+5t^2}+\sqrt{5}\,t}{\sqrt{7}}\right|+C$$

This is a perfectly great answer, but not one you'd usually see in a textbook.

From properties of logarithms,
$$=\frac{1}{\sqrt{5}}\left(\ln\left|\sqrt{7+5t^2}+\sqrt{5}\,t\right|-\ln(\sqrt{7})\right)+C$$
$$=\frac{1}{\sqrt{5}}\ln\left|\sqrt{7+5t^2}+\sqrt{5}\,t\right|\underbrace{-\frac{1}{\sqrt{5}}\ln\sqrt{7}+C}$$
$$=\frac{1}{\sqrt{5}}\ln\left|\sqrt{7+5t^2}+\sqrt{5}\,t\right|+C$$

constant + constant = constant

which is what you'd see in the back of a textbook.

2. Determine $\int \frac{1}{(1+x^2)^3}\,dx$.

Since we have a factor of $1+x^2$ in the denominator, we try

$x = \tan(\theta)$, so $dx = \sec^2(\theta)\,d\theta$

$1+x^2 \Rightarrow 1+\tan^2(\theta) \Rightarrow \sec^2(\theta)$ and we get

$$\int \frac{1}{(1+\tan^2(\theta))^3}\cdot \sec^2(\theta)\,d\theta$$

$$= \int \frac{1}{(\sec^2(\theta))^3}\cdot \sec^2(\theta)\,d\theta$$

$$= \int \frac{1}{\sec^6(\theta)}\cdot \sec^2(\theta)\,d\theta$$

$$= \int \frac{1}{\sec^4(\theta)}\,d\theta$$

$$= \int \cos^4(\theta)\,d\theta$$

$$= \int (\cos^2(\theta))^2\,d\theta$$

$$= \int \left(\tfrac{1}{2}(1+\cos(2\theta))\right)^2 d\theta$$

$$= \int \frac{1}{4}\left(1 + 2\cos(2\theta) + \cos^2(2\theta)\right)d\theta$$

$$= \frac{1}{4}\int 1 + 2\cos(2\theta) + \tfrac{1}{2}(1+\cos(4\theta))\,d\theta$$

$$= \frac{1}{4}\int 1 + 2\cos(2\theta) + \tfrac{1}{2} + \tfrac{1}{2}\cos(4\theta)\,d\theta$$

$$= \frac{1}{4}\int \tfrac{3}{2} + 2\cos(2\theta) + \tfrac{1}{2}\cos(4\theta)\,d\theta$$

$$= \frac{1}{4}\left(\frac{3}{2}\theta + \sin(2\theta) + \frac{1}{8}\sin(4\theta)\right)$$

$$= \frac{3}{8}\theta + \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta) + C$$

Now we need to return to our original variable x.

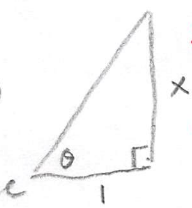
From $x = \tan(\theta)$, we see

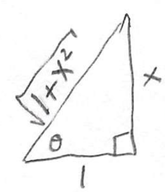
$\theta = \tan^{-1}(x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

For $\sin(2\theta)$, we use the double-angle formula

$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

and now we need $\sin(\theta)$ and $\cos(\theta)$. Again from $x = \tan(\theta)$ and basic trigonometry, we draw the triangle shown at right. Check that $x = \tan(\theta)$ in this triangle. Pythagorean Theorem gives the hypotenuse as $\sqrt{1+x^2}$. So from the second triangle,





* Note that in the triangle, $0<\theta<\frac{\pi}{2}$.

$$\sin(\theta) = \frac{x}{\sqrt{1+x^2}}$$

and

$$\cos(\theta) = \frac{1}{\sqrt{1+x^2}}.\quad \text{Thus } \sin(2\theta) = 2\cdot\frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+x^2}}$$

$$= \frac{2x}{1+x^2}$$

Now, $\sin(4\theta) = 2\sin(2\theta)\cos(2\theta)$

$$= 2\sin(2\theta)\left(\cos^2(\theta) - \sin^2(\theta)\right)$$

$$= 2\cdot\frac{2x}{1+x^2}\cdot\left(\left(\frac{1}{\sqrt{1+x^2}}\right)^2 - \left(\frac{x}{\sqrt{1+x^2}}\right)^2\right)$$

$$= \frac{4x(1-x^2)}{(1+x^2)^2}$$

Finally,

$$\int \frac{1}{(1+x^2)^3}\,dx = \frac{3}{8}\tan^{-1}(x) + \frac{1}{4}\cdot\frac{2x}{1+x^2} + \frac{1}{32}\cdot\frac{4x(1-x^2)}{(1+x^2)^2} + C$$

$$= \frac{3}{8}\tan^{-1}(x) + \frac{x}{2(1+x^2)} + \frac{x(1-x^2)}{8(1+x^2)^2} + C$$