SCC Math 185

Integration by Partial Fractions

We discussed the general idea of integrating with partial fractions on Wednesday. We concluded that the challenge will be to start with an algebraic fraction (a.k.a. rational function) and then to rewrite that fraction as the sum or difference of other, simpler, fractions.

Please google "integration by partial fractions" and find the link to Paul's Online Notes: Calculus II - Partial Fractions. Carefully read his discusson, paying close attention to the table showing "Factor in denominator" and "Term in partial fraction decomposition." Stop reading BEFORE example 1. (If you do not have internet access, then all this information is in your textbook.)

Now turn to page 486 in your textbook and carefully study Example 2. In equation $\boxed{3}$ you'll note how Mr. Stewart uses Paul's table to decompose the algebraic fraction. Mr. Stewart completes the process by solving for A, B, and C. This is the best method to find those values. This method is best because it always works. It does take a bit more effort, but again, it always works.

On p. 487, Mr. Stewart has a NOTE that explains the other method. This is the method that Paul emphasizes and that 10 out of 10 websites that I looked at demonstrate. There is nothing wrong with this method, except that there are situations (including a number in the HW) where this method does not give a complete answer.

Given the choice between a method that <u>always</u> works and one that sometimes works, I'll choose the former everytime! In class (both 185 and 287 – where we do <u>lots</u> of partial fractions) I will always use the method on p. 486.

Let's do a couple of examples!

<u>ex</u> $\displaystyle\int \frac{x^2-x+6}{x^3+3x}\,dx$

Factor the denominator.

$$= \int \frac{x^2-x+6}{x(x^2+3)}\,dx$$

From Paul's table,

$$\frac{x^2-x+6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

Multiply both sides by the common denominator $x(x^2+3)$

$$x^2-x+6 = A(x^2+3)+(Bx+C)x$$
$$x^2-x+6 = Ax^2+3A+Bx^2+Cx$$
$$x^2-x+6 = Ax^2+Bx^2+Cx+3A$$
$$x^2-x+6 = (A+B)x^2+Cx+3A$$

* This equ'n is an <u>identity</u>, so do <u>not</u> add or subtract across the = sign!!

$$x^2 - x + 6 = (A+B)x^2 + Cx + 3A$$

One x^2 on the left

$A+B$ x^2 on the right $\Rightarrow$ $1 = A+B$

$$x^2 - x + 6 = (A+B)x^2 + Cx + 3A$$

-1 x on the left

C x on the right $\Rightarrow$ $-1 = C$

$$x^2 - x + 6 = (A+B)x^2 + Cx + 3A$$

6 is the constant on the left

$3A$ is the constant on the right $\Rightarrow$ $6 = 3A$

So we get the system of equations

$$\begin{aligned} A + B &= 1 \\ C &= -1 \\ 3A &= 6 \end{aligned}$$

and we do whatever algebra it takes to solve for A, B, and C.

Since $3A = 6$, $A = 2$.

Clearly, $C = -1$

Since $A = 2$, $2 + B = 1$

$B = -1$

Now that we have A, B, and C, we get

$$\frac{x^2-x+6}{x(x^2+3)} = \frac{2}{x} + \frac{-x-1}{x^2+3}$$

Thus,

$$\int \frac{x^2-x+6}{x(x^2+3)}\,dx = \int \frac{2}{x} + \frac{-x-1}{x^2+3}\,dx$$

$$= \int \frac{2}{x}\,dx + \int \frac{-x}{x^2+3}\,dx + \int \frac{-1}{x^2+3}\,dx$$

$$= 2\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{2x}{x^2+3}\,dx - \int \frac{1}{x^2+(\sqrt{3})^2}\,dx$$

Let $u = x^2+3$, $du = 2x\,dx$

Formula on p. A89

$$= 2\ln|x| - \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

That was so much fun, let's do another!

ex: $\displaystyle\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}\,dx$

$$\frac{x^2-2x-1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

* LCD is $(x-1)^2(x^2+1)$

$$x^2-2x-1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$$

$$x^2-2x-1 = A(x^3+x-x^2-1) + Bx^2+B + (Cx+D)(x^2-2x+1)$$

$$x^2-2x-1 = Ax^3 - Ax^2 + Ax - A + Bx^2 + B + Cx^3 - 2Cx^2 + Cx + Dx^2 - 2Dx + D$$

$$x^2-2x-1 = Ax^3 + Cx^3 - Ax^2 + Bx^2 - 2Cx^2 + Dx^2 + Ax + Cx - 2Dx - A + B + D$$

$$x^2-2x-1 = (A+C)x^3 + (-A+B-2C+D)x^2 + (A+C-2D)x + (-A+B+D)$$

$0x^3$ on the left

$A+C$ x^3 on the right

$\Longrightarrow$ $A+C=0$

Following the same reasoning, we get

$$\begin{aligned}
&① \quad A \phantom{{}-A+B} + C \phantom{{}+D} = 0\\
&② \quad -A + B - 2C + D = 1\\
&③ \quad A \phantom{{}+B} + C - 2D = -2\\
&④ \quad -A + B \phantom{{}-2C} + D = -1
\end{aligned}$$

③ − ① gives $\quad -2D = -2 \Rightarrow D = 1$

② − ④ gives $\quad -2C = 2 \Rightarrow C = -1$

① gives $\quad A = 1$

Substituting in 4 give

$$-1 + B + 1 = -1 \Rightarrow B = -1$$

So

$$\frac{x^2-2x-1}{(x-1)^2(x^2+1)} = \frac{1}{x-1} + \frac{-1}{(x-1)^2} + \frac{-x+1}{x^2+1}$$

$$\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}dx = \int \frac{1}{x-1}dx - \int \frac{1}{(x-1)^2}dx + \int \frac{-x+1}{x^2+1}dx$$

$$= \ln|x-1| - \int (x-1)^{-2}dx + \int \frac{-x}{x^2+1}dx + \int \frac{1}{x^2+1}dx$$

Let $u = x^2+1$ so $du = 2x\,dx$

$$= \ln|x-1| - \frac{(x-1)^{-1}}{-1} - \frac{1}{2}\int \frac{2x}{x^2+1}dx + \tan^{-1}(x)$$

$$= \ln|x-1| + \frac{1}{x-1} - \frac{1}{2}\ln(x^2+1) + \tan^{-1}(x) + C$$

Your text has many other examples. Paul's Online Notes has <u>6</u> fully worked out and explained examples. Typing "Partial Fractions" into YouTube gave 21,200 results; "Partial Fraction Integration" gave 10,300 results.

- Try some problems from the Sec. 7.4 assignment.
- Begin working on Part I of the assignment from Sec. 7.5. (There is <u>no</u> new material in 7.5. It is just <u>lots</u> of mixed practice.)