Math 185, Calculus II name: ______________________

Sec. 7-7 #23 date: ______________________

The trouble with error estimates is that it is often very difficult to compute four derivatives and obtain a good upper bound K for $\left|f^{(4)}(x)\right|$ by hand. But computer algebra systems have no problem computing $f^{(4)}$ and graphing it, so we can easily find a value for K from a machine graph. This exercise deals with approximations to the integral $I = \int_0^{2\pi} f(x)\,\mathrm{d}x$, where $f(x) = \mathrm{e}^{\cos(x)}$

(a) Use a graph to get a good upper bound for $|f''(x)|$.

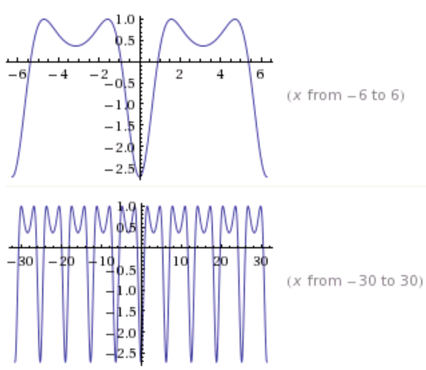
Inputing "second derivative of " to WolframAlpha, we get the following results:

Derivative: Show steps

$$\frac{d^2}{dx^2}\left(e^{\cos(x)}\right) = e^{\cos(x)}\left(\sin^2(x) - \cos(x)\right)$$

second derivative of e^(cos(x))

Plots:



second derivative of e^(cos(x))

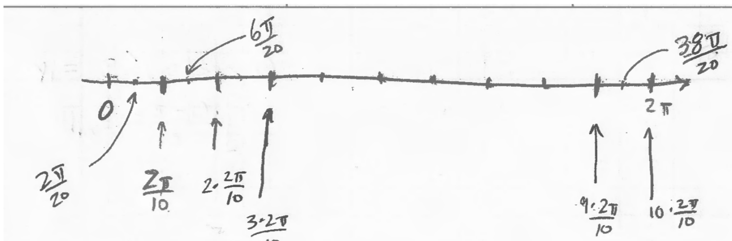
Global minima: Approximate form

$$\min\left\{\frac{\partial^2 e^{\cos(x)}}{\partial x^2}\right\} = -e \text{ at } x = 2\,n\,\pi \text{ for integer } n$$

second derivative of e^(cos(x))

From either the graph or the indicated minimum, we choose $K = |-\mathrm{e}| \approx 2.72$.

(b) Use M_{10} to approximate I.



So midpoints are $x = 1\cdot\frac{2\pi}{20},\ 3\cdot\frac{2\pi}{20},\ 5\cdot\frac{2\pi}{20},\ \dots,\ 19\cdot\frac{2\pi}{20}$

$y_1 = e\text{\^{}}(\cos(x))$

$M_{10} = (2\pi - 0)/10 * (y_1(2\pi/20) + y_1(3 * 2\pi/20) + \cdots + y_1(19\cdot 2\pi/20))$

≈ 7.954926518

(c) Use part (a) to estimate the error in part (b).

$$|E_m| \leq \frac{e(2\pi - 0)^3}{24 * 10^2} \approx 0.2809459949$$

OR

$$|E_m| \leq \frac{2.72(2\pi - 0)^3}{24 \cdot 10^2} \approx 0.2811235752$$

(d) Use the built-in numerical integration capability of your CAS to approximate I.

Definite integral: Fewer digits | More digits

$$\int_0^{2\pi} e^{\cos(x)}\,dx = 2\pi I_0(1) \approx$$

7.954926521012845274513219665329394328161342771816638573400·.
5959553833606081646946669951373572285687 74

$I_n(z)$ is the modified Bessel function of the first kind

definite integral of e^(cos(x)) from x=0 to x=2pi — WolframAlpha

(e) How does the actual error compare with the error estimate in part (c)?

The actual error is $|7.954926521 - 7.954926518| = 0.000000003 = 3 \times 10^{-9}$ which is much less than the 2.8×10^{-1} we found in part (c).

(f) Use a graph to get a good upper bound for $\left|f^{(4)}(x)\right|$.

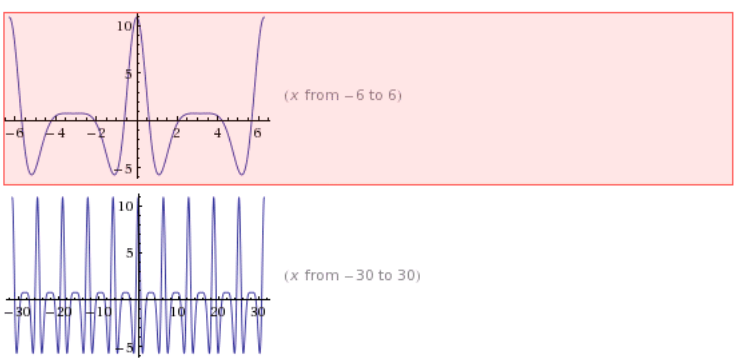
Inputing "fourth derivative of " to WolframAlpha, we get the following results:

Derivative: Show steps

$$\frac{d^4}{dx^4}\left(e^{\cos(x)}\right) = \frac{1}{8}\,e^{\cos(x)}\left(-4\cos(x) + 24\cos(2x) + 12\cos(3x) + \cos(4x) - 1\right)$$

fourth derivative of e^(cos(x)) — WolframAlpha

Plots:



fourth derivative of e^(cos(x)) — WolframAlpha

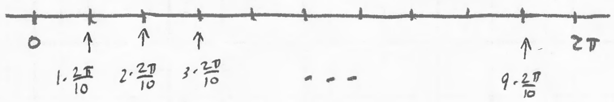
Global maximum: Approximate form

$$\max\left\{\frac{\partial^4 e^{\cos(x)}}{\partial x^4}\right\} = 4e \quad \text{at} \quad x = 0$$

fourth derivative of e^(cos(x)) — WolframAlpha

From either the graph or the indicated maximum, we choose $K = |4e| \approx 10.88$.

(g) Use S_{10} to approximate I.



$Y_1 = e\verb|^|(\cos(x))$

$$S_{10} = ((2\pi/10)/3)\big(Y_1(0) + 4Y_1(1*2\pi/10) + 2Y_1(2*2\pi/10) + \cdots + 4Y_1(9\cdot 2\pi/10) + Y_1(2\pi)\big)$$

$$\approx 7.953789422$$

(h) Use part (f) to estimate the error in part (g).

$$|E_S| \le \frac{4e(2\pi-0)^5}{180\cdot 10^4} \approx 0.0591536177$$

OR

$$|E_S| \le \frac{10.88(2\pi-0)^5}{180\cdot 10^4} \approx 0.0591910075$$

(i) How does the actual error compare with the error estimate in part (h)?

The actual error is $|7.954926521 - 7.953789422| = 0.000000001 = 1\times 10^{-9}$ which is much less than the 5.9×10^{-2} we found in part (c).

(j) How large should n be to guarantee that the size of the error in S_n is less than 0.0001?

We solve

$$\frac{4e(2\pi-0)^5}{180\cdot n^4} \le 0.0001$$

$$\frac{4e(2\pi)^5}{180\cdot 0.0001} \le n^4 \Rightarrow n \ge \sqrt[4]{\frac{4e(2\pi)^5}{180\cdot 0.0001}}$$

$$n \ge 49.3$$

So we take $n \ge 50$ to ensure an error of less than 0.0001.

(10.88 gives the same result.)