The trouble with error estimates is that it is often very difficult to compute four derivatives and obtain a good upper bound $K$ for $|f^{(4)}(x)|$ by hand. But computer algebra systems have no problem computing $f^{(4)}$ and graphing it, so we can easily find a value for $K$ from a machine graph. This exercise deals with approximations to the integral $I = \int_0^{2\pi} f(x) \, dx$, where $f(x) = e^{\cos(x)}$

(a) Use a graph to get a good upper bound for $|f''(x)|$.

Inputing “second derivative of $f$” to WolframAlpha, we get the following results:

From either the graph or the indicated minimum, we choose $K = |-e| \approx 2.72$.

(b) Use $M_{10}$ to approximate $I$.

\[
\begin{align*}
M_{10} &= \frac{(2\pi)}{10} \left[ f\left(\frac{2\pi}{20}\right) + f\left(\frac{3\pi}{20}\right) + \cdots + f\left(\frac{19\pi}{20}\right) \right] \\
&= 7.954926518
\end{align*}
\]
(c) Use part (a) to estimate the error in part (b).

\[ |E_m| \leq \frac{e^{(2\pi-\epsilon)}-e^{2\pi}}{24\times10^{-5}} \approx 0.2809459949 \]

or

\[ |E_m| \leq \frac{2.72 (z_0-\epsilon)\frac{1}{24\times10^{-5}}} \approx 0.2811235752 \]

(d) Use the built-in numerical integration capability of your CAS to approximate \( I \).

\[
\int_0^\infty e^{\cos(x)} \, dx = 2\pi I_0(1) = \]

\[7.95492652101284527451321966532399432816134277181616439573409\]

\[ 59595383050681646566995137357228568774 \]

\( I_0(z) \) is the modified Bessel function of the first kind.

(e) How does the actual error compare with the error estimate in part (c)?

The actual error is \(|7.954926521 - 7.954926518| = 0.000000003 = 3 \times 10^{-9}\) which is much less than the \(2.8 \times 10^{-1}\) we found in part (c).

(f) Use a graph to get a good upper bound for \(|f^{(4)}(x)|\).

Inputing “fourth derivative of” to WolframAlpha, we get the following results:

From either the graph or the indicated maximum, we choose \( K = |4e| \approx 10.88 \).
(g) Use \( S_{10} \) to approximate \( I \).

\[
S_{10} = \left( \frac{2\pi}{10} \right) \left( y_{1,5} + 4y_{1,2} + 2y_{1,10} \right) \\
\approx 7.953789422
\]

(h) Use part (f) to estimate the error in part (g).

\[
\left| E_{S} \right| \leq \frac{4e(2\pi-0)^5}{180 \cdot 10^4} \approx 0.0591536177
\]

\[
\left| E_{S} \right| \leq \frac{10.89(2\pi-0)^5}{180 \cdot 10^4} \approx 0.0591910075
\]

(i) How does the actual error compare with the error estimate in part (h)?

The actual error is \( |7.954926521 - 7.953789422| = 0.000000001 = 1 \times 10^{-9} \) which is much less than the \( 5.9 \times 10^{-2} \) we found in part (c).

(j) How large should \( n \) be to guarantee that the size of the error in \( S_n \) is less than 0.0001?

We solve

\[
\frac{4e(2\pi-0)^5}{180 \cdot n^4} \leq 0.0001
\]

\[
\frac{4e(2\pi)^5}{180 \cdot 0.0001} \leq n^4 \Rightarrow n \geq \sqrt[4]{\frac{4e(2\pi)^5}{180 \cdot 0.0001}}
\]

\[
n \geq 49.3
\]

So we take \( n \geq 50 \) to ensure an error of less than 0.0001. (10.88 gives the same result.)