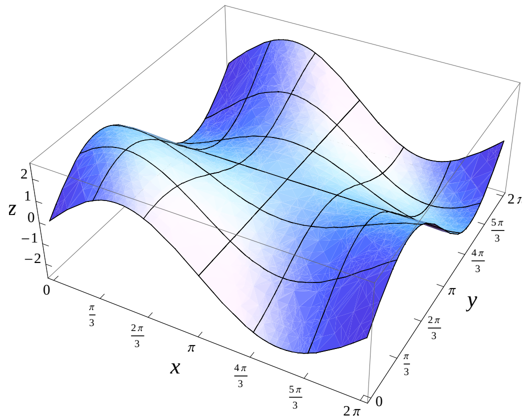
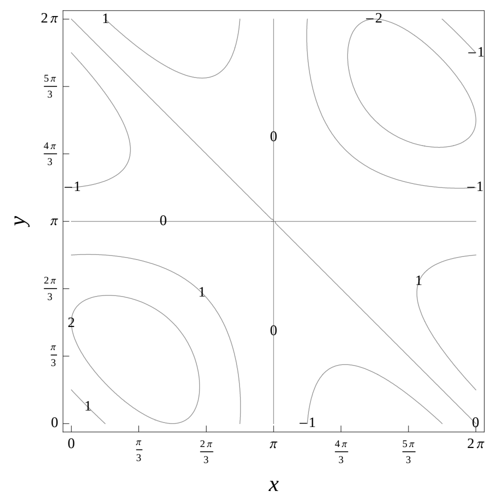
Math 280, Intermediate Calculus name: ____________________

Sec. 14.7 Maximum and Minimum Values date: ____________________

23. Use a graph and/or level curves to estimate the local maximum and minimum values and saddle point(s) of the function. Then use calculus to find these values precisely.

$$f(x, y) = \sin x + \sin y + \sin(x + y), \quad 0 \le x \le 2\pi, \quad 0 \le y \le 2\pi$$





From the 3D-plot and the contour plot, it seems that there is a local maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, a saddle point at (π, π), and a local minimum at $\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right)$.

$$\begin{aligned} f_x &= \cos x + \cos(x + y) \\ f_y &= \cos y + \cos(x + y) \\ f_{xx} &= -\sin x - \sin(x + y) \\ f_{yy} &= -\sin y - \sin(x + y) \\ f_{xy} &= -\sin(x + y) \end{aligned}$$

If we set $f_x = 0$ and $f_y = 0$, and subtract, we get $\cos x = \cos y$. Thus $x = y$ or $x = 2\pi - y$.

If we substitute $x = y$ into $f_x = 0$, we get

$$\cos x + \cos 2x = 0$$

and since

$$\cos 2x = 2\cos^2 x - 1$$

we substitute to get

$$\begin{aligned} 2\cos^2 x + \cos x - 1 &= 0 \\ (2\cos x - 1)(\cos x + 1) &= 0 \end{aligned}$$

so

$$\cos x = \frac{1}{2} \text{ or } \cos x = -1$$

and thus

$$x = \frac{\pi}{3}, \; x = \frac{5\pi}{3} \text{ or } x = \pi$$

From these x-values, we get the critical points

$$(\pi, \pi),\ \left(\frac{\pi}{3}, \frac{\pi}{3}\right), \text{ and } \left(\frac{5\pi}{3}, \frac{5\pi}{3}\right)$$

Now we apply the Second Partials Test to these critical points.

$$\begin{aligned} D(x, y) &= \sin x \sin y + \sin x \sin(x+y) + \sin y \sin(x+y) \\ D(\pi, \pi) &= 0 \end{aligned}$$

and thus the SPT does not apply. However, along the line $y = x$ we have

$$\begin{aligned} f(x, x) &= 2 \sin x + \sin 2x \\ &= 2 \sin x + 2 \sin x \cos x \\ &= 2 \sin x (1 + \cos x) \end{aligned}$$

Now $f(x, x) > 0$ for $0 < x < \pi$. Also, $f(x, x) < 0$ for $\pi < x < 2\pi$. So in any neighborhood of (π, π) there are points where the function is positive and points where the function is negative. Thus, there is a saddle point at (π, π).

Furthermore,

$$D\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{9}{4} > 0$$

and

$$f_{xx}\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = -\sqrt{3} < 0$$

So $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$ is a local maximum.

Finally,

$$D\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) = \frac{9}{4} > 0$$

and

$$f_{xx}\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) = \sqrt{3} > 0$$

So $f\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) = -\frac{3\sqrt{3}}{2}$ is a local minimum.