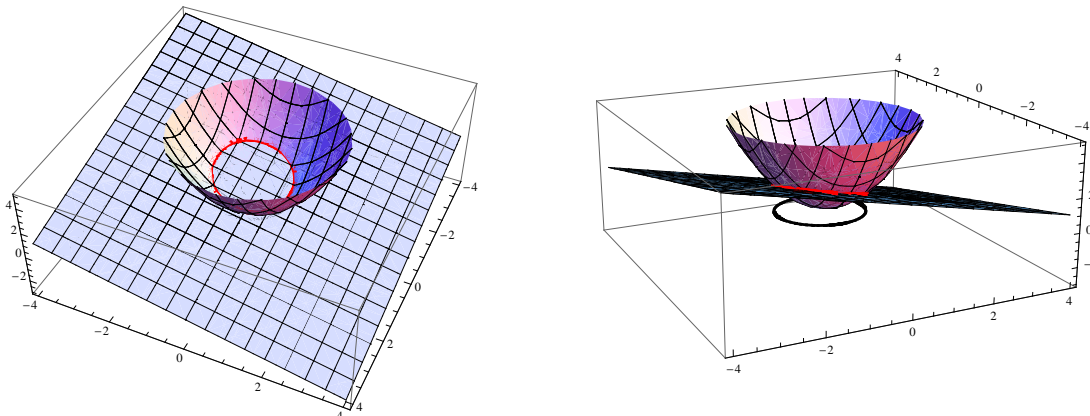


41. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on the ellipse that are nearest to and farthest from the origin.



Here, the two constraints are $g(x, y, z) = x + y + 2z - 2$ and $h(x, y, z) = x^2 + y^2 - z$. Any critical point that we find during the Lagrange multiplier process will satisfy both of these constraints, so we actually don't need to find an explicit equation for the ellipse that is their intersection.

Suppose that (x, y, z) is any point that satisfies both of the constraints (and hence is on the ellipse.)

Then the distance from (x, y, z) to the origin is given by $\sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2}$. This expression (and its partial derivatives) would be cumbersome to work with, so we will find the the extrema of the *square* of the distance. Thus, our objective function is

$$f(x, y, z) = x^2 + y^2 + z^2$$

and

$$\begin{aligned} \nabla f &= \langle 2x, 2y, 2z \rangle \\ \lambda \nabla g &= \langle \lambda, \lambda, 2\lambda \rangle \\ \mu \nabla h &= \langle 2\mu x, 2\mu y, -\mu \rangle \end{aligned}$$

Thus the system we need to solve for (x, y, z) is

$$2x = \lambda + 2\mu x \tag{1}$$

$$2y = \lambda + 2\mu y \tag{2}$$

$$2z = 2\lambda - \mu \tag{3}$$

$$x + y + 2z = 2 \tag{4}$$

$$x^2 + y^2 - z = 0 \tag{5}$$

Subtracting (2) from (1) and factoring gives

$$2(x - y) = 2\mu(x - y)$$

so $\mu = 1$ whenever $x \neq y$. Substituting $\mu = 1$ into (1) gives us $\lambda = 0$ and substituting $\mu = 1$ and $\lambda = 0$ into (3) gives us $2z = -1$ and thus $z = -\frac{1}{2}$. Substituting $z = -\frac{1}{2}$ into (4) and (5) gives us

$$x + y - 3 = 0$$

$$x^2 + y^2 + \frac{1}{2} = 0$$

however, $x^2 + y^2 + \frac{1}{2} = 0$ has no solution. Thus we must have $x = y$.

Since we now know $x = y$, (4) and (5) become

$$\begin{aligned}2x + 2z &= 2 \\2x^2 - z &= 0\end{aligned}$$

so

$$\begin{aligned}z &= 1 - x \\z &= 2x^2\end{aligned}$$

Combining these together gives us $2x^2 = 1 - x$, so $2x^2 + x - 1 = 0$ which has solutions $x = \frac{1}{2}$ and $x = -1$.

Further substitution yields the critical points $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(-1, -1, 2)$. Substituting these into our objective function gives us

$$\begin{aligned}f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) &= \frac{3}{4} \\f(-1, -1, 2) &= 6\end{aligned}$$

Thus minimum distance of $\frac{\sqrt{3}}{2}$ occurs at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and the maximum distance of $\sqrt{6}$ occurs at $(-1, -1, 2)$.